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Lyapunov-type inequalities for a higher order fractional differential equation with fractional integral boundary conditions

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Abstract. New Lyapunov-type inequalities are derived for the fractional boundary value problem

$$\begin{aligned} &D_a^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \\ &u(a) = u'(a) = \cdots = u^{(n-2)}(a) = 0, \quad u(b) = I_a^\alpha (hu)(b), \end{aligned}$$

where $n \in \mathbb{N}$, $n \geq 2$, $n-1 < \alpha < n$, D_a^α denotes the Riemann–Liouville fractional derivative of order α, I_a^α denotes the Riemann–Liouville fractional integral of order α, and $q, h \in C([a,b];\mathbb{R})$. As an application, we obtain numerical approximations of lower bound for the eigenvalues of corresponding equations.

Keywords: Lyapunov-type inequality, fractional boundary value problem, eigenvalue.

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1 Introduction

In this paper, we obtain new Lyapunov-type inequalities for the fractional boundary value problem

$$D_a^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \tag{1.1}$$

$$u(a) = u'(a) = \cdots = u^{(n-2)}(a) = 0, \quad u(b) = I_a^\alpha (hu)(b), \tag{1.2}$$

where $n \in \mathbb{N}$, $n \geq 2$, $n-1 < \alpha < n$, D_a^α denotes the Riemann–Liouville fractional derivative of order α, I_a^α denotes the Riemann–Liouville fractional integral of order α, and $q, h \in C([a,b];\mathbb{R})$. We use a Green's function approach that consists in transforming the fractional boundary value problem (1.1)–(1.2) into an equivalent integral form and then find the maximum of the

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modulus of its Green's function. In the case $n = 2$, we obtain a generalization of the Lyapunov-type inequality established by Ferreira in [12]. In the case $n \geq 3$, the study of the Green's function is more complex. The obtained Lyapunov-type inequalities in such case involve the solution of a certain nonlinear equation that belongs to the interval $\left(0, \left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}\right)$. Some numerical results are presented in order to estimate such solution for different values of α. As an application of our obtained Lyapunov-type inequalities, we present some numerical approximations of lower bound for the eigenvalues of corresponding equations.

Let us start by describing some historical backgrounds about Lyapunov inequality and some related works. In the late 19th century, the mathematician A. M. Lyapunov established the following result (see [27]).

Theorem 1.1. *If the boundary value problem*

$$\begin{cases} u''(t) + q(t)u(t) = 0, \quad a < t < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1.3}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}. \tag{1.4}$$

Inequality (1.4) is known as "Lyapunov inequality". It proved to be very useful in various problems in connection with differential equations, including oscillation theory, asymptotic theory, eigenvalue problems, disconjugacy, etc. For more details, we refer the reader to [3–5, 8, 14, 16, 17, 30, 33, 35, 37] and references therein.

In [17], Hartman and Wintner proved that if the boundary value problem (1.3) has a nontrivial solution, then

$$\int_a^b (s-a)(b-s)q^+(s)\, ds > b-a, \tag{1.5}$$

where

$$q^+(s) = \max\{q(s), 0\}, \quad s \in [a, b].$$

Using the fact that

$$\max_{a \leq s \leq b} (s-a)(b-s) = \frac{(b-a)^2}{4},$$

Lyapunov inequality (1.4) follows immediately from inequality (1.5). Many other generalizations and extensions of inequality (1.4) exist in the literature, see for instance [6, 9–11, 15, 16, 19, 26, 29, 31, 32, 36, 38, 39] and references therein.

Due to the positive impact of fractional calculus on several applied sciences (see for instance [25]), several authors investigated Lyapunov type inequalities for various classes of fractional boundary value problems. The first work in this direction is due to Ferreira [12], where he considered the fractional boundary value problem

$$\begin{cases} D_a^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1.6}$$

where $(a, b) \in \mathbb{R}^2$, $a < b$, $\alpha \in (1, 2)$, $q : [a, b] \to \mathbb{R}$ is a continuous function, and D_a^α is the Riemann–Liouville fractional derivative operator of order α. The main result obtained in [12] is the following fractional version of Theorem 1.1.

Theorem 1.2. *If the fractional boundary value problem* (1.6) *has a nontrivial solution, then*

$$\int_a^b |q(s)|\,ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}, \tag{1.7}$$

where Γ *is the Gamma function.*

Observe that (1.4) can be deduced from Theorem 1.2 by passing to the limit as $\alpha \to 2$ in (1.7).

For other results on Lyapunov-type inequalities for fractional boundary value problems, we refer the reader to [1,2,7,13,20–24,28,34] and references therein.

Before stating and proving the main results in this work, some preliminaries are needed. This is the aim of the next section.

2 Preliminaries

We start this section by briefly recalling some concepts on fractional calculus.

Let I be a certain interval in $\mathbb{R}$. We denote by $AC(I;\mathbb{R})$ the space of real valued and absolutely continuous functions on I. For $n = 1, 2, \ldots$, we denote by $AC^n(I;\mathbb{R})$ the space of real valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on I with $f^{(n-1)} \in AC(I;\mathbb{R})$, that is

$$AC^n(I;\mathbb{R}) = \left\{ f : I \to \mathbb{R} \text{ such that } D^{n-1}f \in AC(I;\mathbb{R}) \ \left(D = \frac{d}{dx}\right)\right\}.$$

Cleraly, we have $AC^1(I;\mathbb{R}) = AC(I;\mathbb{R})$.

Definition 2.1 (see [25])**.** Let $f \in L^1((a,b);\mathbb{R})$, where $(a,b) \in \mathbb{R}^2$, $a < b$. The Riemann–Liouville fractional integral of order $\alpha > 0$ of f is defined by

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}\,ds, \quad \text{a.e. } t \in [a,b].$$

Definition 2.2 (see [25])**.** Let $\alpha > 0$ and n be the smallest integer greater or equal than α. The Riemann–Liouville fractional derivative of order α of a function $f : [a,b] \to \mathbb{R}$, where $(a,b) \in \mathbb{R}^2$, $a < b$, is defined by

$$(D_a^\alpha f)(t) = \left(\frac{d}{dt}\right)^n I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,ds, \quad \text{a.e. } t \in [a,b],$$

provided that the right-hand side is defined almost everywhere on $[a,b]$.

Let $\alpha > 0$ and n be the smallest integer greater or equal than α. By $AC^\alpha([a,b];\mathbb{R})$, where $(a,b) \in \mathbb{R}^2$, $a < b$, we denote the set of all functions $f : [a,b] \to \mathbb{R}$ that have the representation:

$$f(t) = \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha-n+1+i)}(t-a)^{\alpha-n+i} + I_a^\alpha \varphi(t), \quad \text{a.e. } t \in [a,b], \tag{2.1}$$

where $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$ and $\varphi \in L^1((a,b);\mathbb{R})$.

The next lemma provides a necessary and sufficient condition for the existence of $D_a^\alpha f$ for $f \in L^1((a,b);\mathbb{R})$.

Lemma 2.3 (see [18]). *Let $\alpha > 0$ and n be the smallest integer greater or equal than α. Let $f \in L^1((a,b);\mathbb{R})$, where $(a,b) \in \mathbb{R}^2$, $a < b$. Then $D_a^\alpha f(t)$ exists almost everywhere on $[a,b]$ if and only if $f \in AC^\alpha([a,b];\mathbb{R})$, that is, f has the representation (2.1). In such a case, we have*

$$(D_a^\alpha f)(t) = \varphi(t), \quad a.e.\ t \in [a,b].$$

Lemma 2.4. *Let $y \in C([a,b];\mathbb{R})$. If $v \in C([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ satisfies*

$$D_a^\alpha v(t) = y(t), \qquad a < t < b,$$

where $n \in \mathbb{N}$, $n \geq 2$, $n-1 < \alpha < n$, then

$$c_0 = v(a) = 0,$$

with c_0 is the constant that appears in the representation (2.1).

Proof. By Lemma 2.3, we have

$$v(t) = \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - n + 1 + i)} (t-a)^{\alpha - n + i} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds, \qquad t \in (a,b].$$

Since v is continuous on $[a,b]$, we have

$$\lim_{t \to a^+} v(t) = v(a).$$

On the other hand, observe that

$$\lim_{t \to a^+} v(t) = \lim_{t \to a^+} \frac{c_0 (t-a)^{\alpha - n}}{\Gamma(\alpha - n + 1)}.$$

Since $\alpha - n < 0$, we deduce that $c_0 = v(a) = 0$. □

Lemma 2.5. *Let $y \in C([a,b];\mathbb{R})$. If $v \in C^{n-2}([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ is a solution of the fractional boundary value problem*

$$D_a^\alpha v(t) = y(t), \quad a < t < b,$$
$$v(a) = v'(a) = \cdots = v^{(n-2)}(a) = 0, \quad v(b) = 0,$$

where $n \in \mathbb{N}$, $n \geq 2$, $n-1 < \alpha < n$, then

$$v(t) = \int_a^b -G(t,s) y(s)\, ds, \qquad a < t < b,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \leq t \leq s \leq b. \end{cases} \tag{2.2}$$

Proof. Let $v \in C^{n-2}([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ be a solution of the considered fractional boundary value problem. By Lemmas 2.3 and 2.4, we obtain

$$v(t) = \sum_{i=1}^{n-1} d_i (t-a)^{\alpha - i} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds,$$

where d_i are some constants. Next, we have

$$v'(t) = \sum_{i=1}^{n-1} d_i(\alpha - i)(t-a)^{\alpha-i-1} + \frac{1}{\Gamma(\alpha)} \int_a^t (\alpha-1)(t-s)^{\alpha-2} y(s)\, ds.$$

The boundary condition $v'(a) = 0$ yields $d_{n-1} = 0$. Continuing this process, we obtain

$$d_i = 0, \quad i = 2, 3, \ldots, n-2.$$

Therefore,

$$v(t) = d_1(t-a)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds.$$

By the condition $v(b) = 0$, we get

$$d_1(b-a)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} y(s)\, ds = 0.$$

Then

$$d_1 = \frac{-1}{\Gamma(\alpha)(b-a)^{\alpha-1}} \int_a^b (b-s)^{\alpha-1} y(s)\, ds.$$

Hence, we have

$$v(t) = \frac{-(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}} \int_a^b (b-s)^{\alpha-1} y(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds,$$

which yields the desired result. □

Lemma 2.6. *If $u \in C^{n-2}([a,b];\mathbb{R}) \cap AC^{\alpha}([a,b];\mathbb{R})$ is a solution of (1.1)–(1.2), then*

$$u(t) = \int_a^b G(t,s)(q(s)+h(s))u(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)u(s)\, ds, \qquad a \le t \le b,$$

where G is the Green's function defined by (2.2).

Proof. Let $u \in C^{n-2}([a,b];\mathbb{R}) \cap AC^{\alpha}([a,b];\mathbb{R})$ be a solution of (1.1)–(1.2). Let us introduce the function

$$v(t) = u(t) - I_a^{\alpha}(hu)(t), \qquad a \le t \le b, \tag{2.3}$$

that is,

$$v(t) = u(t) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)u(s)\, ds, \qquad a \le t \le b.$$

Observe that for all $t \in (a,b)$, we have

$$\begin{aligned} D_a^{\alpha} v(t) &= D_a^{\alpha} u(t) - D_a^{\alpha} I_a^{\alpha}(hu)(t) \\ &= D_a^{\alpha} u(t) - h(t)u(t). \end{aligned}$$

Therefore, using (1.1) we obtain

$$\begin{aligned} D_a^{\alpha} v(t) &= D_a^{\alpha} u(t) - h(t)u(t) \\ &= -q(t)u(t) - h(t)u(t) \\ &= -(q(t)+h(t))u(t), \end{aligned}$$

that is,

$$D_a^\alpha v(t) = -(q(t)+h(t))u(t), \qquad a < t < b. \tag{2.4}$$

On the other hand, using (2.3) we obtain

$$\begin{aligned} v'(t) &= u'(t) - \frac{(\alpha-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} h(s)u(s)\,ds \\ v''(t) &= u''(t) - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-3} h(s)u(s)\,ds \\ &\vdots \\ v^{(n-2)}(t) &= u^{(n-2)}(t) - \frac{(\alpha-1)\cdots(\alpha-n+2)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-n+1} h(s)u(s)\,ds, \end{aligned}$$

for all $t \in [a,b]$. Therefore, from (1.2) we obtain

$$v(a) = v'(a) = \cdots = v^{(n-2)}(a) = 0, \qquad v(b) = 0. \tag{2.5}$$

Then $v \in C^{n-2}([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ is a solution of the fractional boundary value problem (2.4)–(2.5). Next, using Lemma 2.5 we have

$$v(t) = \int_a^b G(t,s)(q(s)+h(s))u(s)\,ds, \qquad a \le t \le b.$$

Therefore, by (2.3) we obtain

$$u(t) = \int_a^b G(t,s)(q(s)+h(s))u(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)u(s)\,ds, \qquad a \le t \le b,$$

which proves the desired result. □

3 Estimates of the Green's function

In this section, we provide estimates of the Green's function G defined by (2.2) in both cases $n=2$ and $n \ge 3$.

Let us start with the case $n=2$. We have the following result established by Ferreira [12].

Lemma 3.1. *Let $n=2$. The Green's function G defined by (2.2) satisfies the following conditions:*

(i) $G(t,s) \ge 0$, *for all* $a \le t, s \le b$.

(ii) $\displaystyle\max_{a\le t\le b} G(t,s) = G(s,s) = \frac{1}{\Gamma(\alpha)} \frac{(s-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}$, *for all* $a \le s \le b$.

(iii) $\displaystyle\max_{a\le s\le b} G(s,s) = \frac{1}{\Gamma(\alpha)} \left(\frac{b-a}{4}\right)^{\alpha-1}$.

The next lemma provides an estimate of the Green's function G in the case $n \ge 3$.

Lemma 3.2. *Let $n \in \mathbb{N}$, $n \ge 3$. Then*

(i) $G(t,s) \ge 0$, *for all* $a \le t, s \le b$.

(ii) For all $t \in [a,b]$, we have

$$G(t,s) \leq G(s^*,s) = \frac{(s-a)^{\alpha-1}(b-s)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{\alpha-2}}, \qquad a < s < b,$$

where

$$s^* = \frac{s - a\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}}{1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}}.$$

Proof. It can be easily seen that

$$G(t,s) \geq 0, \qquad a \leq t,s \leq b.$$

Let $s \in (a,b)$ be fixed. For $s \leq t \leq b$, we have

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\left(\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}\right).$$

Differentiating with respect to t, we obtain

$$\partial_t G(t,s) = \frac{(\alpha-1)(t-a)^{\alpha-2}}{\Gamma(\alpha)}\left(\left(\frac{b-s}{b-a}\right)^{\alpha-1} - \left(1-\frac{s-a}{t-a}\right)^{\alpha-2}\right).$$

Observe that

$$\partial_t G(t,s) = 0 \Longleftrightarrow t = s^*.$$

On the other hand, we have

$$s^* - a = \frac{s-a}{1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}} > 0, \qquad s \in (a,b)$$

and

$$b - s^* = \frac{b-s}{1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{1}{\alpha-2}}\right] > 0, \qquad s \in (a,b).$$

Therefore, for all $s \in (a,b)$, we have $s^* \in (a,b)$. Moreover, for given $s \in (a,b)$, we have $G(t,s)$ arrives at maximum at s^*, when $s \leq t$. This together with the fact that $G(t,s)$ is increasing on $s > t$, we obtain that (ii) holds. □

Remark 3.3. Observe that in the case $n = 2$, that is, $1 < \alpha < 2$, we have $s^* < a$. Therefore, the estimates for $G(t,s)$ for $n \geq 3$ given in Lemma 3.2 cannot cover those for $n = 2$ given in Lemma 3.1.

Remark 3.4. A simple computation yields

$$\lim_{s \to a^+} G(s^*,s) = \lim_{s \to b^-} G(s^*,s) = 0.$$

By Remark (3.4), the function $(a,b) \ni s \mapsto G(s^*,s)$ can be extended to a continuous function $\varphi : [a,b] \ni s \mapsto \varphi(s)$, where

$$\varphi(s) = \begin{cases} G(s^*,s) & \text{if } a < s < b, \\ 0 & \text{if } s \in \{a,b\}. \end{cases} \tag{3.1}$$

Therefore, there exists a certain $\overline{s} \in (a,b)$ such that

$$\varphi(\overline{s}) = \max\{\varphi(s) : a \leq s \leq b\} = \max\{\varphi(s) : a < s < b\}. \tag{3.2}$$

Using the change of variable $z = \frac{b-s}{b-a}$, $a < s < b$, from (3.2) we obtain

$$\varphi(\overline{s}) = \max\{\mu(z) : 0 < z < 1\}, \tag{3.3}$$

where

$$\mu(z) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \frac{z^{\alpha-1}(1-z)^{\alpha-1}}{\left(1 - z^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}}, \qquad 0 < z < 1. \tag{3.4}$$

Differentiating with respect to z, we obtain

$$\mu'(z) = \frac{(\alpha-1)(b-a)^{\alpha-1}}{\Gamma(\alpha)} z^{\alpha-1}(1-z)^{\alpha-1}\left(1 - z^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2} \nu(z), \qquad 0 < z < 1,$$

where

$$\nu(z) = \frac{z^{\frac{2\alpha-3}{\alpha-2}} - 2z + 1}{z(1-z)\left(1 - z^{\frac{\alpha-1}{\alpha-2}}\right)}, \qquad 0 < z < 1.$$

Clearly, for all $0 < z < 1$, we have

$$\text{sign}(\mu'(z)) = \text{sign}(\nu(z)) = \text{sign}(P(z)),$$

where

$$P(z) = z^{\frac{2\alpha-3}{\alpha-2}} - 2z + 1, \qquad 0 < z < 1.$$

Differentiating with respect to z, we obtain

$$P'(z) = \left(\frac{2\alpha-3}{\alpha-2}\right) z^{\frac{\alpha-1}{\alpha-2}} - 2, \qquad 0 < z < 1.$$

Further, we have

$$P'(z) = 0 \Longleftrightarrow z = \left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}.$$

Moreover, we have $P'(z) \leq 0$ for $z \in \left(0, \left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}\right]$ and $P'(z) \geq 0$ for $z \in \left[\left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}, 1\right)$. Therefore, we have

$$P\left(\left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}\right) < \lim_{z\to 1^-} P(z) = 0.$$

Since

$$1 = \lim_{z\to 0^+} P(z) > 0,$$

there exists a unique $z_\alpha \in \left(0, \left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}\right)$ such that

$$P(z_\alpha) = 0.$$

Hence, we obtain

$$\operatorname{sign}(\mu'(z)) = \begin{cases} + & \text{if } 0 < z \le z_\alpha, \\ - & \text{if } z_\alpha < z < 1, \end{cases}$$

which yields from (3.3) that

$$\varphi(\bar{s}) = \mu(z_\alpha) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \frac{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}{\left(1 - z_\alpha^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}}.$$

From the above analysis and Lemma 3.2, we deduce the following result.

Lemma 3.5. *Let $n \in \mathbb{N}$, $n \ge 3$. Then*

$$\max_{a \le s \le b} G(s^*, s) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \frac{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}{\left(1 - z_\alpha^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}},$$

where z_α is the unique zero of the nonlinear equation

$$z^{\frac{2\alpha-3}{\alpha-2}} - 2z + 1 = 0$$

in the interval $\left(0, \left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}\right)$.

Tables 3.1 and 3.2 provide numerical values of z_α for different values of α and n. The numerical results are obtained using the bisection method implemented in Matlab.

α	17/8	9/4	19/8	5/2	21/8	11/4	23/8
z_α	0.5004	0.5086	0.5246	0.5436	0.5633	0.5825	0.6008

Table 3.1: Values of z_α for $\alpha \in (2, 3)$

α	33/8	17/4	35/8	9/2	37/8	19/4	39/8
z_α	0.7289	0.7376	0.7457	0.7534	0.7606	0.7675	0.7740

Table 3.2: Values of z_α for $\alpha \in (4, 5)$

Figure 3.1 shows the graph of functions $y = \mu(z)$ (normalized) and $z = z_\alpha$ for $\alpha = \frac{5}{2}$, where μ is defined by (3.4). Observe that the function μ attains its maximum at $z = z_\alpha$, which confirm the above theoretical analysis.

4 Lyapunov-type inequalities

We distinguish two cases.

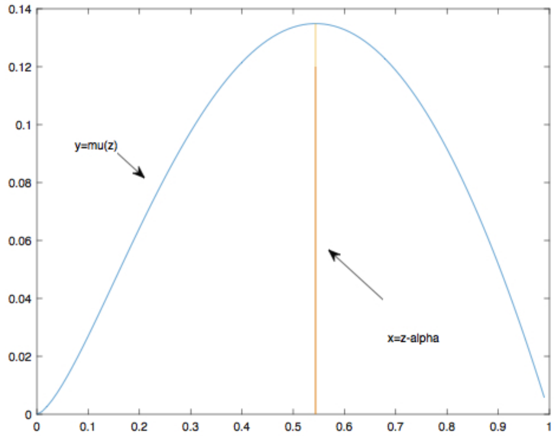

Figure 3.1: *Graph of functions $y = \mu(z)$ and $z = z_\alpha$ for $\alpha = 5/2$*

4.1 The case $n = 2$

In this case, problem (1.1)–(1.2) reduces to

$$D_a^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b \tag{4.1}$$

$$u(a) = 0, \quad u(b) = I_a^\alpha (hu)(b), \tag{4.2}$$

where $1 < \alpha < 2$ and $q, h \in C([a,b];\mathbb{R})$.

We have the following Hartman–Wintner-type inequality for the fractional boundary value problem (4.1)–(4.2).

Theorem 4.1. *If $u \in C([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ is a nontrivial solution of the fractional boundary value problem* (4.1)–(4.2)*, then*

$$\int_a^b (b-s)^{\alpha-1} \left[(s-a)^{\alpha-1} |q(s) + h(s)| + (b-a)^{\alpha-1} |h(s)| \right] ds \geq \Gamma(\alpha)(b-a)^{\alpha-1}. \tag{4.3}$$

Proof. Let $u \in C([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ be a nontrivial solution of the fractional boundary value problem (4.1)–(4.2). Using Lemma 2.6, we have

$$u(t) = \int_a^b G(t,s)(q(s) + h(s))u(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) u(s)\, ds, \qquad a \leq t \leq b. \tag{4.4}$$

Let

$$\|u\| = \max\{|u(t)| : a \leq t \leq b\}, \qquad u \in C([a,b];\mathbb{R}). \tag{4.5}$$

From (4.4), we get

$$|u(t)| \leq \left(\int_a^b |G(t,s)||q(s) + h(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} |h(s)|\, ds \right) \|u\|, \qquad a \leq t \leq b.$$

Using Lemma 3.1 (ii), we obtain

$$\|u\| \leq \left(\int_a^b |G(s,s)||q(s) + h(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} |h(s)|\, ds \right) \|u\|$$

Since u is nontrivial, we have $\|u\| > 0$. Therefore,

$$\begin{aligned} 1 &\leq \int_a^b |G(s,s)||q(s)+h(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|h(s)|\,ds \\ &= \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-1}}\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-1}|q(s)+h(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|h(s)|\,ds, \end{aligned}$$

which yields

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-1}|q(s)+h(s)|\,ds + \int_a^b (b-a)^{\alpha-1}(b-s)^{\alpha-1}|h(s)|\,ds \geq \Gamma(\alpha)(b-a)^{\alpha-1},$$

and the desired inequality (4.3) follows. □

The following Lyapunov-type inequality for the fractional boundary value problem (4.1)–(4.2) holds.

Theorem 4.2. *If $u \in C([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ is a nontrivial solution of the fractional boundary value problem* (4.1)–(4.2)*, then*

$$\int_a^b \left(|q(s)+h(s)| + 4^{\alpha-1}|h(s)|\right)\,ds \geq \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{4.6}$$

Proof. Let $u \in C([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ be a nontrivial solution of the fractional boundary value problem (4.1)–(4.2). Following the proof of Theorem 4.1 and using (4.4), we have

$$|u(t)| \leq \left(\int_a^b |G(t,s)||q(s)+h(s)|\,ds + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b |h(s)|\,ds\right)\|u\|, \qquad a \leq t \leq b.$$

Using Lemma 3.1 (iii), we obtain

$$\|u\| \leq \left(\frac{1}{\Gamma(\alpha)}\left(\frac{b-a}{4}\right)^{\alpha-1}\int_a^b |q(s)+h(s)|\,ds + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b |h(s)|\,ds\right)\|u\|.$$

Since u is nontrivial, we have $\|u\| > 0$. Therefore,

$$\frac{1}{\Gamma(\alpha)}\left(\frac{b-a}{4}\right)^{\alpha-1}\int_a^b |q(s)+h(s)|\,ds + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b |h(s)|\,ds \geq 1,$$

which yields the desired inequality (4.6). □

4.2 The case $n \geq 3$

We have the following Hartman–Wintner-type inequality for the fractional boundary value problem (1.1)–(1.2), in the case $n \geq 3$.

Theorem 4.3. *Let $n \in \mathbb{N}$ with $n \geq 3$. If $u \in C^{n-2}([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ is a nontrivial solution of* (1.1)–(1.2)*, then*

$$\begin{aligned} \frac{1}{(b-a)^{\alpha-1}}\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} &|q(s)+h(s)|\,ds \\ &+ \int_a^b (b-s)^{\alpha-1}|h(s)|\,ds \geq \Gamma(\alpha). \end{aligned} \tag{4.7}$$

Proof. Inequality (4.7) follows from Lemma (ii) 3.2 and by using similar argument as in the proof Theorem 4.1. □

The following Lyapunov-type inequality for the fractional boundary value problem (1.1)–(1.2), in the case $n \geq 3$, holds.

Theorem 4.4. *Let $n \in \mathbb{N}$ with $n \geq 3$. If $u \in C^{n-2}([a,b];\mathbb{R}) \cap AC^{\alpha}([a,b];\mathbb{R})$ is a nontrivial solution of (1.1)–(1.2), then*

$$\int_a^b \left(|q(s)+h(s)| + \frac{\left(1 - z_\alpha^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}} |h(s)| \right) ds \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}} \frac{\left(1 - z_\alpha^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}, \tag{4.8}$$

where z_α is the unique zero of the nonlinear equation

$$z^{\frac{2\alpha-3}{\alpha-2}} - 2z + 1 = 0$$

in the interval $\left(0, \left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}\right)$.

Proof. Inequality (4.8) follows immediately from inequality (4.7) and Lemma 3.5. □

5 The case $h \equiv 0$

If $h \equiv 0$, problem (1.1)–(1.2) reduces to

$$D_a^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b \tag{5.1}$$

$$u(a) = u'(a) = \cdots = u^{(n-2)}(a) = 0, \quad u(b) = 0, \tag{5.2}$$

where $n \in \mathbb{N}$, $n \geq 2$, $n-1 < \alpha < n$, and $q \in C([a,b];\mathbb{R})$.

Taking $h \equiv 0$ in Theorem 4.3, we obtain the following Hartman–Wintner-type inequality for the fractional boundary value problem (5.1)–(5.2), in the case $n = 2$.

Corollary 5.1. *Let $n = 2$. If $u \in C([a,b];\mathbb{R}) \cap AC^{\alpha}([a,b];\mathbb{R})$ is a nontrivial solution of the fractional boundary value problem (5.1)-(5.2), then*

$$\int_a^b (b-s)^{\alpha-1}(s-a)^{\alpha-1}|q(s)|\, ds \geq \Gamma(\alpha)(b-a)^{\alpha-1}.$$

Remark 5.2. Taking $h \equiv 0$ in (4.6), we obtain the result of Ferreira [12] given by Theorem 1.2.

Taking $h \equiv 0$ in Theorem 4.7, we obtain the following Hartman-Wintner-type inequality for the fractional boundary value problem (5.1)–(5.2), in the case $n \geq 3$.

Corollary 5.3. *Let $n \in \mathbb{N}$ with $n \geq 3$. If $u \in C^{n-2}([a,b];\mathbb{R}) \cap AC^{\alpha}([a,b];\mathbb{R})$ is a nontrivial solution of (5.1)–(5.2), then*

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-1} \left[1 - \left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} |q(s)|\, ds \geq \Gamma(\alpha)(b-a)^{\alpha-1}. \tag{5.3}$$

Taking $h \equiv 0$ in Theorem 4.8, we obtain the following Lyapunov-type inequality for the fractional boundary value problem (5.1)–(5.2), in the case $n \geq 3$.

Corollary 5.4. *Let $n \in \mathbb{N}$ with $n \geq 3$. If $u \in C^{n-2}([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$ is a nontrivial solution of (5.1)–(5.2), then*

$$\int_a^b |q(s)|\,ds \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}} \frac{\left(1 - z_\alpha^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}},$$

where z_α is the unique zero of the nonlinear equation

$$z^{\frac{2\alpha-3}{\alpha-2}} - 2z + 1 = 0$$

in the interval $\left(0, \left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}\right)$.

6 Applications to eigenvalue problems

In this section, we present some applications of the obtained results to eigenvalue problems. More precisely, we provide lower bound for the eigenvalues of certain nonlocal boundary value problems.

We say that a scalar λ is an eigenvalue of the fractional boundary value problem

$$D_a^\alpha u(t) = \lambda u(t), \quad 0 < t < 1 \tag{6.1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = 0, \tag{6.2}$$

where $n \in \mathbb{N}$, $n \geq 2$, $n-1 < \alpha < n$, iff (6.1)–(6.2) admits at least a nontrivial solution (eigenvector) $u_\lambda \in C^{n-2}([a,b];\mathbb{R}) \cap AC^\alpha([a,b];\mathbb{R})$.

Corollary 6.1. *Let $n = 2$. If λ is an eigenvalue of the fractional boundary value problem (6.1)–(6.2), then*

$$|\lambda| \geq C_\alpha := \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}.$$

Proof. Let λ be an eigenvalue of the fractional boundary value problem (6.1)–(6.2). Then problem (6.1)–(6.2) admits at least one eigenvector u_λ. Using Corollary 5.1 with $q \equiv \lambda$ and $(a,b) = (0,1)$, we obtain

$$|\lambda| \int_0^1 (1-s)^{\alpha-1} s^{\alpha-1}\,ds \geq \Gamma(\alpha).$$

Note that

$$\int_0^1 (1-s)^{\alpha-1} s^{\alpha-1}\,ds = B(\alpha,\alpha),$$

where B is the beta function. Using the relation

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we obtain

$$\int_0^1 (1-s)^{\alpha-1} s^{\alpha-1}\,ds = \frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)}.$$

Therefore, we have

$$|\lambda| \geq \frac{\Gamma(\alpha)}{B(\alpha,\alpha)} = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)},$$

and the desired result follows. □

Figure 6.1 shows the behavior of C_α with respect to $\alpha \in (1,2)$.

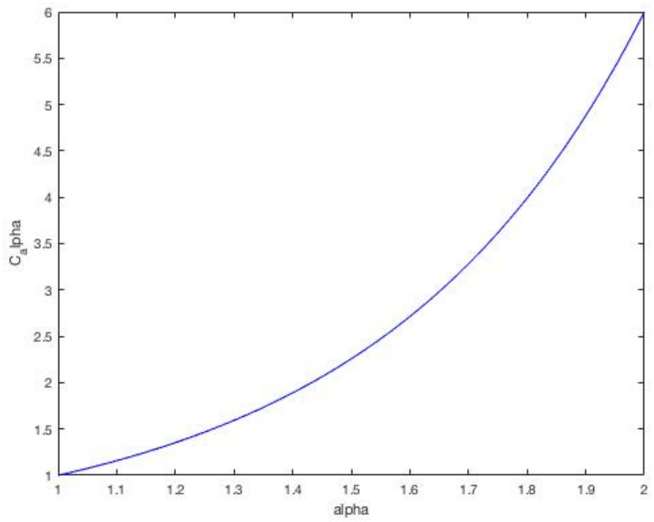

Figure 6.1: *Graph of function C_α for $\alpha \in (1,2)$*

Corollary 6.2. *Let $n \in \mathbb{N}$ with $n \geq 3$. If λ is an eigenvalue of the fractional boundary value problem (6.1)–(6.2), then*

$$|\lambda| \geq D_\alpha := \Gamma(\alpha)\left(\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\left[1-(1-s)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} ds\right)^{-1}.$$

Proof. The result follows using Corollary 5.3 with $q \equiv \lambda$ and $(a,b) = (0,1)$, and a similar argument as in the proof of Corollary 6.1. □

Table 6.1 provides numerical approximations of D_α for different values of $\alpha \in [2.2,3]$. The numerical values are obtained using numerical integrations via Matlab.

α	2.2	2.3	2.4	2.5	2.6	2.9	3
D_α	9.0130	10.9139	13.1348	15.7385	18.8010	31.7499	37.7636

Table 6.1: Numerical values of D_α for $\alpha \in [2.2,3]$

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