Research Article

Multiple Positive Solutions for Quadratic Integral Equations of Fractional Order

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Received 26 March 2017; Accepted 6 September 2017; Published 15 October 2017

Academic Editor: Hua Su

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In this paper, the existence of multiple positive solutions for a class of quadratic integral equation of fractional order is obtained, by utilizing Avery-Henderson and Leggett-Williams multiple fixed point theorems on cones. A example is given to illustrate the applicability of our results. We believe that this is a first result concerning the existence of multiple solutions for such quadratic integral equation of fractional order.

1. Introduction and Preliminaries

Recently, there has been great interest for many authors to study quadratic functional integral equations, which has become one of the most attractive and interesting research areas of integral equations and functional integral equations. There is large literature on this topic. We refer the reader to [1–6] for some of very recent results. In fact, as noted in some earlier literature (see, e.g., [5] and references therein), the nonlinear quadratic functional integral equations have been applied to, for example, the theory of radiative transfer, kinetic theory of gases, the theory of neutron transport, the traffic theory, plasma physics, and numerous branches of mathematical physics.

On the other hand, due to the fact that fractional differential and integral equations have recently been extensively applied in various areas of engineering, mathematics, physics and bioengineering, and other applied sciences, there has been a significant development in fractional integral equations in recent years. Many authors especially have focused on the existence and qualitative properties of solutions for quadratic integral equations of fractional order such as

\[
x(t) = f(t, x(t)) + g(t, x(t)) \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s, x(s)) \, ds,
\]

\(\alpha > 0,\)

and several important variants of (1), and obtain substantial results on this topic (cf. [1–3, 5–9]). However, to the best of our knowledge, it seems that there are no results concerning the existence of multiple solutions for (1) and its variants. That is the main goal of this work. It seems that this is a first result concerning the existence of multiple solutions for such quadratic integral equation of fractional order.

Stimulated by the above works, we aim to investigate the existence of multiple positive solutions for the following fractional order quadratic integral equation:

\[
x(t) = f \left[ t, x \left( \lambda(t) \right) \right] + g \left[ t, x \left( \mu(t) \right) \right] \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h \left[ s, x \left( \nu(s) \right) \right] \, ds,
\]

\(t \in [0, 1], \ \alpha > 0.\)

See Section 2 for the hypotheses on the involved functions.
Throughout the rest of this paper, if there is no special statement, we denote by \(\mathbb{R}\) the set of real numbers, and by \(\mathcal{Lip}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^+)\) the set of all functions \(f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+\) satisfying that there exists a constant \(L_f > 0\) such that
\[
|f(t,x) - f(t,y)| \leq L_f |x-y|, \quad t \in \mathbb{R}, \ x, y \in \mathbb{R}. \tag{3}
\]

Next, let us recall some notations about cones and two fixed point theorem. For more details, we refer the reader to [10, 11].

Let \(X\) be a real Banach space. A closed convex set \(K\) in \(X\) is called a cone if the following conditions are satisfied:
\begin{enumerate}[(i)]
  
  \item If \(x \in K\), then \(\lambda x \in K\) for any \(\lambda \geq 0\);
  
  \item If \(x \in K\) and \(-x \in K\), then \(x = 0\).
\end{enumerate}

A nonnegative continuous functional \(\psi\) is said to be a concave continuous mapping on \(K\) if \(\psi\) is continuous and
\[
\psi(\mu x + (1 - \mu) y) \geq \mu \psi(x) + (1 - \mu) \psi(y), \quad x, y \in K, \ \mu \in [0, 1]. \tag{4}
\]

Letting \(c_1, c_2, c_3\) be three positive constants and \(\phi\) be a nonnegative continuous functional on \(K\), we denote
\[
K_{c_1} = \{y \in K : \|y\| < c_1\}, \quad \overline{K}_{c_1} = \{y \in K : \|y\| \leq c_1\},
K(\phi, c_1) = \{x \in K : \phi(x) < c_1\}, \quad \overline{K}(\phi, c_1) = \{x \in K : \phi(x) \leq c_1\},
\partial K(\phi, c_1) = \{x \in K : \phi(x) = c_1\},
K(\phi, c_2, c_3) = \{y \in K : c_2 \leq \phi(y), \|y\| \leq c_3\}. \tag{5}
\]

In addition, we say that \(\phi\) is increasing on \(K\) if \(\phi(x) \geq \phi(y)\) for all \(x, y \in K\) with \(x - y \in K\).

The following two theorems are the well-known Avery-Wilson multiple fixed point theorem and Leggett-Williams multiple fixed point theorem, respectively.

**Lemma 1** (see [10]). Let \(K\) be a cone in a real Banach space \(X\), \(c_1\) be a positive constant, \(\Phi : \overline{K}_{c_1} \to \overline{K}_{c_1}\) be a completely continuous mapping, and \(\psi\) be a concave nonnegative continuous functional on \(K\) with \(\psi(u) \leq \|u\|\) for all \(u \in K_{c_1}\). Suppose that there exist three constants \(c_1, c_2, c_3\) with \(0 < c_1 < c_2 < c_3 < c_4\) such that
\begin{enumerate}[(i)]
  
  \item \(\{u \in K(\phi, c_1) : \phi(u) > c_2\} \neq \emptyset\), and \(\psi(\Phi u) > c_2\) for all \(u \in K(\phi, c_1)\);
  
  \item \(\|\Phi u\| < c_4\) for all \(u \in K_{c_1}\);
  
  \item \(\psi(\Phi u) > c_2\) for all \(u \in K(\phi, c_2, c_3)\) with \(\|\Phi u\| > c_3\).
\end{enumerate}

Then \(\Phi\) has at least three fixed points \(u_1, u_2, u_3\) in \(K_{c_1}\). Furthermore, \(\|u_1\| < \|u_2\|, \quad \psi(u_2) < c_2 < \psi(u_3)\).

2. Main Results

Firstly, we list some assumptions:
\begin{enumerate}[(H1)]
  
  \item \(f, g, h \in C_1([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)\) and \(f, g \in \mathcal{Lip}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^+)\).
  
  \item \(\lambda, \mu, \nu \in C([0, 1], \mathbb{R}^+)\). Moreover, \(\lambda'(t), \mu'(t) > 0\) and \(\nu'(t) > 1\).
  
  \item \(\inf_{t \in [0, 1], x \geq 0} f(t, x) > 0, \sup_{t \in [0, 1], x \geq 0} f(t, x) < +\infty, \sup_{t \in [0, 1], x \geq 0} g(t, x) < +\infty, \) and
\[
\lim_{r \to +\infty} \frac{\sup_{t \in [0, 1], x \geq 0} h(t, x)}{r} = 0. \tag{9}
\]
  
  \item There exist \(0 < b < c\) such that
\[
\inf_{x \geq 0} f(1, x) + \inf_{x \geq 0} g(1, x) \inf_{v'(t) \leq 1} h(t, c) \frac{[1 - v^{-1}(1)]^\alpha}{\alpha \Gamma(\alpha)} > c, \tag{10}
\]
\[
\sup_{t \in [0, 1], x \geq 0} f(t, x) + \sup_{t \in [0, 1], x \geq 0} g(t, x) \sup_{t \in [0, 1], x \geq 0} h(t, x) \frac{1}{\alpha \Gamma(\alpha)} < b.
\]
\end{enumerate}
By a solution of (2) we mean a function \( x \in C[0, T] \) satisfying the equation, where

\[
T = \max \left\{ \max_{t \in [0,1]} \lambda(t), \max_{t \in [0,1]} \mu(t), \max_{t \in [0,1]} \nu(t) \right\}.
\]  

(11)

Now, we are ready to present our main result.

**Theorem 3.** Let (H1)–(H4) hold. Then, there exists \( L^* > 0 \) such that (2) has at least two nonnegative solutions provided that

\[
L_f < 1 \quad \text{and} \quad L_g < L^*.
\]

**Proof.** By using (H3), we can choose \( \sigma \in (0,1) \) satisfying

\[
\inf_{t \in [0,1], x \geq 0} f(t, x) \geq \sigma \left( \sup_{t \in [0,1], x \geq 0} f(t, x) + \sup_{t \in [0,1], x \leq 0} h(t, x) \cdot \alpha \Gamma(\alpha) \right).
\]  

(12)

Let

\[
K = \left\{ x \in C[0, T] : x(t) = x(1) \text{ for every } t \in [1, T], \min_{t \in [0,1]} x(t) \geq \sigma \|x\| \right\}.
\]

(13)

It is not difficult to verify that \( K \) is a cone in \( C[0, T] \). Let \( \Psi(\lambda) = \lambda \Psi(1) \) for every \( \lambda \in (0,1) \).

\[
(\Psi\lambda)(t) = x_{\lambda}(1),
\]

(17)

It is easy to see that \( A_{\lambda} \Omega \subset \Omega \). For \( x, z \in \Omega \) and \( r \in [0,1] \), there holds

\[
\left| (A_{\lambda}r)(t) - (A_{\lambda}z)(t) \right| \leq L_f \|x - z\| + \frac{L_g}{\alpha \Gamma(\alpha)} \sup_{t \in [0,1], x \leq 0} h(t, x) \|x - z\|,
\]

(18)

which yields that

\[
\|A_{\lambda}x - A_{\lambda}z\| = \max_{t \in [0, T]} \left| (A_{\lambda}x)(t) - (A_{\lambda}z)(t) \right| \leq L_f \|x - z\| + \frac{L_g}{\alpha \Gamma(\alpha)} \sup_{t \in [0,1], x \leq 0} h(t, x) \|x - z\|,
\]

(19)

Let

\[
L^* = \frac{\alpha \Gamma(\alpha) \left( 1 - L_f \right)}{\sup_{t \in [0,1], x \leq 0} h(t, x)}.
\]

(20)

Then, noting that \( L_f < 1 \), \( A_{\lambda} \) has a unique fixed point \( x_{\lambda} \in \Omega \) provided that \( L_g < L^* \).

**Step 2.** Now, define an operator \( A \) on \( \overline{K(\phi, c)} \) by

\[
(Ay)(t) = x_y(t) = \begin{cases} f \left[ t, x_y(\lambda(t)) \right] + g \left[ t, x_y(\mu(t)) \right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h[s, y(v(s))] ds, & y \in K(\phi, c), \ t \in [0,1], \\ (Ay)(1), & y \in K(\phi, c), \ t \in [1, T]. \end{cases}
\]

(21)
Noting that, for every \( t \in [0,1] \) and \( y \in K(\varphi,c) \), by (12), we have
\[
(Ay)(t) \geq \inf_{t \in [0,1], x \geq 0} f(t,x) \geq \sigma \left( \sup_{t \in [0,1], x \geq 0} f(t,x) + \frac{\sup_{t \in [0,1], x \geq 0} g(t,x) \cdot \sup_{t \in [0,1], x \geq 0} h(t,x)}{\alpha \Gamma(\alpha)} \right)
\]
\[
\geq \sigma \|Ay\|
\]
which yields that \( A(K(\varphi,c)) \subset K \).

Next, let us show that \( A : K(\varphi,c) \to K \) is completely continuous. Let \( y_n \to y \) in \( K(\varphi,c) \). For \( t \in [0,1] \), we have
\[
\|Ay_n - Ay\| = \max_{t \in [0,T]} |(Ay_n)(t) - (Ay)(t)| = \max_{t \in [0,1]} |(Ay_n)(t) - (Ay)(t)|
\]
\[
\leq \frac{\alpha \Gamma(\alpha) \sup_{t \in [0,1], x \geq 0} g(t,x) \cdot \sup_{t \in [0,1]} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s,y_n(s)) \, ds - g[t_1, (Ay)(\mu(t_1))] + \sup_{t \in [0,1], x \geq 0} g(t,x) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s,y(s)) \, ds}
\]
where
\[
\alpha \Gamma(\alpha) > \alpha \Gamma(\alpha) L_f + L_g \sup_{t \in [0,1], x \in [0,c/\sigma]} h(t,x)
\]
(25)
since \( L_g < L^* \). Combining (24) with the fact \( h \) is uniformly continuous on \([0,1] \times [0,c/\sigma] \) and \( y_n(t) \) uniformly converges to \( y(t) \) on \([0,T] \), we conclude that \( Ay_n \to Ay \) in \( K \). It suffices to show \( A(K(\varphi,c)) \) is precompact. It follows from the above proof that \( \{Ay(t) : y \in K(\varphi,c) \} \) is uniformly bounded on \([0,T] \). For every \( y \in K(\varphi,c) \) and \( t_1, t_2 \in [0,1] \) with \( t_1 \leq t_2 \), we have
\[
|((Ay)(t_1) - (Ay)(t_2))| \leq |f[t_1, (Ay)(\lambda(t_1))]| - f[t_2, (Ay)(\lambda(t_2))] + \left| g[t_1, (Ay)(\mu(t_1))] \right|
\]
\[
\cdot \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h[s, y(\nu(s))] \, ds - g[t_2, (Ay)(\mu(t_2))]
\]
(26)
On the other hand, we have
\[
\left| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds - \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right|
\leq\int_0^{t_1} \left| \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \right| ds
+ \frac{t_1^\alpha (t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)}
\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |s^{\alpha-1} - (t_2 - t_1 + s)^{\alpha-1}| ds
+ \frac{1}{\alpha \Gamma(\alpha)} (t_2 - t_1)^\alpha.
\]

By Lebesgue’s dominated convergence theorem, as \(t_2 - t_1 \to 0\),
\[
\int_0^{t_1} |s^{\alpha-1} - (t_2 - t_1 + s)^{\alpha-1}| ds
\leq \int_0^1 |s^{\alpha-1} - (t_2 - t_1 + s)^{\alpha-1}| ds \to 0.
\]
Combining this with the fact that \(f, g\) are uniformly continuous on compact sets, we conclude that, for every \(\varepsilon > 0\), there exists \(\delta' > 0\) such that, for all \(t_1, t_2 \in [0, 1]\) with \(t_1 \leq t_2\) and \(|t_1 - t_2| < \delta'\), and \(y \in K(\varphi, c)\), which yields that, for all \(t_1, t_2 \in [0, 1]\) with \(t_1 \leq t_2\) and \(|t_1 - t_2| < \delta'\), there holds
\[
\left. \begin{array}{l}
\sup_{y \in K(\varphi, c)} |(Ay)(t_1) - (Ay)(t_2)| \\
\leq L_f \cdot \sup_{y \in K(\varphi, c)} |(Ay)(\lambda(t_1)) - (Ay)(\lambda(t_2))| \\
+ L_g \cdot \sup_{y \in K(\varphi, c)} \left( \sup_{\lambda \in K(\varphi, c)} |(Ay)(\mu(t_1)) - (Ay)(\mu(t_2))| + \varepsilon \right).
\end{array} \right\}
\]

Then, we have
\[
\lim_{t_1 \to t_1} \left( \sup_{y \in K(\varphi, c)} |(Ay)(t_1) - (Ay)(t_2)| \right) \leq L_f
\]
\[
\lim_{t_1 \to t_1} \left( \sup_{y \in K(\varphi, c)} \left( \sup_{\lambda \in K(\varphi, c)} |(Ay)(\lambda(t_1)) - (Ay)(\lambda(t_2))| \right) \right)
+ \frac{L_g \cdot \sup_{y \in K(\varphi, c)} \left( \sup_{\lambda \in K(\varphi, c)} |(Ay)(\mu(t_1)) - (Ay)(\mu(t_2))| \right)}{\alpha \Gamma(\alpha)} + \varepsilon,
\]
where
\[
\lim_{t_1 \to t_1} \left( \sup_{y \in K(\varphi, c)} |(Ay)(t_1) - (Ay)(t_2)| \right) = \inf_{\delta > 0} \sup_{\lambda \in K(\varphi, c)} \left( \sup_{y \in K(\varphi, c)} |(Ay)(t_1) - (Ay)(t_2)| \right),
\]

are similarly defined. Noting that \(\lambda'(t), \mu'(t) > 0\) for all \(t \in [0, 1]\), we conclude
\[
\lim_{t_1 \to t_1} \left( \sup_{y \in K(\varphi, c)} |(Ay)(t_1) - (Ay)(t_2)| \right)
= \lim_{t_1 \to t_1} \left( \sup_{y \in K(\varphi, c)} \left( \sup_{\lambda \in K(\varphi, c)} |(Ay)(\lambda(t_1)) - (Ay)(\lambda(t_2))| \right) \right)
= \lim_{t_1 \to t_1} \left( \sup_{y \in K(\varphi, c)} |(Ay)(\mu(t_1)) - (Ay)(\mu(t_2))| \right),
\]

holds for all \(t_1, t_2 \in [0, 1]\) with \(t_1 \leq t_2\) and \(|t_1 - t_2| < \delta'\), and \(y \in K(\varphi, c)\), which yields that, for all \(t_1, t_2 \in [0, 1]\) with \(t_1 \leq t_2\) and \(|t_1 - t_2| < \delta'\), there holds
\[
\sup_{y \in K(\varphi, c)} \left| (Ay)(t_1) - (Ay)(t_2) \right|
\leq L_f \cdot \sup_{y \in K(\varphi, c)} \left| (Ay)(\lambda(t_1)) - (Ay)(\lambda(t_2)) \right|
+ L_g \cdot \sup_{y \in K(\varphi, c)} \left( \sup_{\lambda \in K(\varphi, c)} \left| (Ay)(\mu(t_1)) - (Ay)(\mu(t_2)) \right| + \varepsilon \right).
\]
and so we have
\[
\limsup_{t_2-t_1 \to 0} \left( \sup_{x \in K(\varphi, c)} \left| (Ay)(t_1) - (Ay)(t_2) \right| \right) 
\leq \frac{\alpha \Gamma(\alpha) \epsilon}{\alpha \Gamma(\alpha) - \alpha \Gamma(\alpha) L_f - \alpha \Gamma(\alpha) L_g \sup_{t \in [0,1], x \in [c, \sigma]} h(t, x)}.
\]
(35)

Thus, \((Ay)(t) : y \in K(\varphi, c)\) is equicontinuous on \([0, 1]\).

Then, it is not difficult to obtain that \((Ay)(t) : y \in K(\varphi, c)\)
is equicontinuous on \([0, T]\). This proves that \(A(K(\varphi, c))\) is precompact, and thus \(A : K(\varphi, c) \to K\) is completely continuous.

**Step 3.** It remains to verify the assumptions (i)–(iii) of Lemma 1. For every \(y \in \partial K(\varphi, c)\), noting that \(y(1) = c\) and \(y(t) = c\) for all \(t \in [1, T]\), by (H4), we have
\[
\varphi(Ay) = (Ay)(1) = f[1, (Ay)(\lambda(1))] + g[1, (Ay)(\mu(t))] \int_0^1 \left(1-s\right)^{\alpha-1} h[s, y(\nu(s))] \, ds
\geq \inf_{x \in \partial K(\varphi, c)} f(1, x) + \inf_{x \in \partial K(\varphi, c)} g(1, x) \int_0^1 \left(1-s\right)^{\alpha-1} h(s, c) \, ds
\geq \inf_{x \in \partial K(\varphi, c)} f(1, x) + \inf_{x \in \partial K(\varphi, c)} g(1, x) \inf_{r \in [1, 1 \leq s \leq 1]} h(s, c)
\cdot \left[1 - r^{-1} \right]^{\alpha} > c.
\]
(36)

For every \(y \in \partial K(\varphi, c)\), noting that \(y(t) \leq b\) for all \(t \in [0, T]\), again by (H4), we have
\[
\varphi(Ay) = \sup_{t \in [1, 1]} \left( f[t, (Ay)(\lambda(t))] + g[t, (Ay)(\mu(t))] \cdot \int_0^t \left(1-s\right)^{\alpha-1} h[s, y(\nu(s))] \, ds \right)
\leq \sup_{t \in [0, 1]} \sup_{x \in [0, c]} f(t, x) + \sup_{t \in [0, 1]} \sup_{x \in [0, b]} g(t, x)
\cdot \sup_{t \in [0, 1]} \sup_{x \in [0, b]} h(t, x)
\leq \sup_{t \in [0, 1]} f(t, x) + \sup_{t \in [0, 1]} g(t, x) \sup_{x \in [0, \rho]} h(t, x)
\cdot \frac{1}{\alpha} < b.
\]
(37)

Then, by Lemma 1, we conclude that \(A\) has at least two fixed points \(x_1, x_2 \in K(\varphi, c)\). Thus,
\[
x_i(t) = f[t, x_i(\lambda(t))]
+ g[t, x_i(\mu(t))] \int_0^t (t-s)^{\alpha-1} h[s, x_i(\nu(s))] \, ds,
\]
(39)

That is, \(x_1, x_2\) are two nonnegative solutions for (2). \(\square\)

**Remark 4.** Due to the influence of fractional term \((t-s)^{\alpha-1}\) in (2), it becomes more difficult to verify the assumptions of Lemma 1. In addition, we have tried to consider the case of \(\lambda(t) \leq 1\), where it seems that the approach used here cannot be applied to study (2). We leave it for further research.

By using Lemma 2, we can obtain the existence of three nonnegative solutions to (2).

**Corollary 5.** Let (H1)–(H4) hold and the first inequality in (H4) is strengthened to
\[
\inf_{x \in \partial K(\varphi, c)} f(1, x) + \inf_{x \in \partial K(\varphi, c)} g(1, x) \inf_{r \in [1, 1 \leq s \leq 1]} h(s, c)
\cdot \left[1 - r^{-1} \right]^{\alpha} > c.
\]
(40)

Then, there exists \(L_* > 0\) such that (2) has at least three nonnegative solutions provided that \(L_f < 1\) and \(L_g < L_*\).

**Proof.** By using (H3), we can choose \(\rho \in (0, 1)\) such that
\[
\inf_{t \in [0, 1], x \in [0, \rho]} f(t, x) \geq \rho \left( \sup_{t \in [0, 1], x \in [0, \rho]} f(t, x) + \frac{\sup_{t \in [0, 1], x \in [0, \rho]} g(t, x) \sup_{t \in [0, 1], x \in [0, \rho]} h(t, x)}{\alpha} \right),
\]
(41)
\[
\sup_{t \in [0, 1], x \in [0, \rho]} f(t, x) \sup_{t \in [0, 1], x \in [0, \rho]} g(t, x) \sup_{t \in [0, 1], x \in [0, \rho]} h(t, x) \leq \frac{c}{\rho}.
\]
(42)

Let \(T\) be the same as in Theorem 3 and
\[
K = \left\{ x \in C [0, T] : x(t) = x(1) \text{ for every } t \right\}.
\]
(43)
Moreover, denote $\Omega$ and $A_y$ ($y \in \overline{K}_{c_4}$) be the same as in Theorem 3, and
\[ c_1 = b, \]
\[ c_2 = c, \]
\[ c_3 = c_4 = \frac{c}{\rho}. \] (44)

Then, by a similar proof to Step 1 of Theorem 3, we can prove that $A_y$ has a unique fixed point $x_y \in \Omega$ provided that $L_g < L_*$, where
\[ L_* = \frac{\alpha \Gamma(\alpha) \left( 1 - L_f \right)}{\sup_{x \in [0,1], t \in [0,\rho/\rho]} h(t, x)}. \] (45)

Now, we can define
\[
(Ay)(t) = x_y(t) = \begin{cases} f \left[ t, x_y(\lambda(t)) \right] + g \left[ t, x_y(\mu(t)) \right] \\
(\text{44}) \end{cases}
\]
$$
\begin{align*}
(Ay)(t) &= f \left[ t, x_y(\lambda(t)) \right] + g \left[ t, x_y(\mu(t)) \right] \\
&= f \left[ t, x_y(\lambda(t)) \right] + g \left[ t, x_y(\mu(t)) \right] \\
&= f \left[ t, x_y(\lambda(t)) \right] + g \left[ t, x_y(\mu(t)) \right].
\end{align*}
$$

Analogously to the proof of Theorem 3, we can also show that $A : \overline{K}_{c_4} \to K$ is completely continuous. Let
\[
\psi(u) = u(1), \quad u \in K.
\] (47)

Obviously, $\psi$ is a concave nonnegative continuous functional on $K$ and $\psi(u) \leq\| u \|$ for every $y \in \overline{K}_{c_4}$, by (42), we have
\[
\|Ay\| \leq \sup_{t \in [0,1]} \left\{ f \left[ t, x_y(\lambda(t)) \right] + g \left[ t, x_y(\mu(t)) \right] \right\} \\
\cdot \left\{ \int_0^1 (t-s)^{\alpha-1} \Gamma(\alpha) h(s, y((s))) ds \right\} \\
\leq \sup_{t \in [0,1]} \left\{ f(t, x) + g(t, x) \right\} \\
\cdot \sup_{t \in [0,1], x \geq 0} h(t, x) \int_0^1 (t-s)^{\alpha-1} \Gamma(\alpha) ds \\
\leq \sup_{t \in [0,1], x \geq 0} f(t, x) \\
+ \frac{\sup_{t \in [0,1], x \geq 0} g(t, x) \sup_{t \in [0,1], x \in [0,\rho]} h(t, x)}{\alpha \Gamma(\alpha)} \\
< \frac{c}{\rho} = c_4,
\] (48)

which means that $A$ maps $\overline{K}_{c_4}$ to $\overline{K}_{c_4}$. Similarly, by (H4), we have
\[
\|Ay\| \leq \sup_{t \in [0,1], x \geq 0} f(t, x) \\
+ \frac{\sup_{t \in [0,1], x \geq 0} g(t, x) \sup_{t \in [0,1], x \in [0,\rho]} h(t, x)}{\alpha \Gamma(\alpha)} \\
< b = c_1,
\] (49)

that is, the condition (ii) of Lemma 2 holds. Moreover, the condition (iii) of Lemma 2 follows from the definition of $K$ and $c_4$. It remains to show that the condition (i) of Lemma 2 holds. For every $y \in K(\psi, c_2, c_3)$, noting that $y(1) \geq c_2 = c$ and thus $y(t) \geq c$ for all $t \in [1, T]$, by (40), we have
\[
\psi(Ay) = (Ay)(1) \\
= f \left[ 1, x_y(\lambda(1)) \right] + g \left[ 1, x_y(\mu(1)) \right] \\
\geq \inf_{x \geq 0} f(1, x) + \inf_{x \geq 0} g(1, x) \\
\cdot \sup_{t \in [0,1], x \in [0,\rho]} h(t, x) \int_0^1 (1-s)^{\alpha-1} \Gamma(\alpha) ds \\
\geq \inf_{x \geq 0} f(1, x) + \inf_{x \geq 0} g(1, x) \frac{1}{\alpha \Gamma(\alpha)} > c.
\] (50)

Then, by Lemma 2, $A$ has at least three fixed points in $\overline{K}_{c_4}$, and thus (2) has at least three nonnegative solutions. \hfill \square

Next, we present an example to show the applicability of our results.

Example 6. For $t \in [0,1]$ and $x \geq 0$, let $\alpha = 1/2$:
\[
\lambda(t) = \mu(t) = t, \\
\nu(t) = 2t, \\
f(t, x) = 1 + \frac{1}{4} \sin^2 (tx), \\
g(t, x) = 1 + \beta e^{-x^2}, \\
h(t, x) = \begin{cases} \sqrt{x}, & 0 \leq x \leq 10, \\
\sqrt{2}y - 1 - \sqrt{2}y \sqrt{10}, & 10 < x < 20, \\
y \sqrt{x}, & x \geq 20,
\end{cases}
\] (51)

where $\beta, y > 0$ are two constants.
It is not difficult to show that (H1) holds with $L_f \leq 1/2$ and $L_g \leq \beta \sqrt{2} e^{-1/2}$. Moreover, it is easy to verify that (H2) and (H3) hold. Letting $b = 10$ and $c = 20$, we have

$$\inf_{x \geq 0} f(1, x) + \inf_{x \geq 0} g(1, x) \inf_{r \geq 1, \mu \leq 1, x \geq \infty} h(s, x) \cdot \left[ \frac{1 - y^{-1}(1)}{\alpha \Gamma(\alpha)} \right] \geq 1 + y \sqrt{20} \cdot \sqrt{\frac{2}{\pi}} = 1 + y \sqrt{\frac{20}{\pi}} \geq 20 = c,$$

provided that $y > \sqrt{19\pi/40}$, and

$$\sup_{t \in [0,1], x \geq 0} f(t, x) + \sup_{t \in [0,1], x \geq 0} g(t, x) \sup_{t \in [0,1], x \in [0, b]} h(t, x) \leq \frac{5}{4} \frac{1 + \beta}{\sqrt{\pi/2}} < 10 = b,$$

provided that $\beta < 1$. Then, by Corollary 5, (2) has at least three nonnegative solutions provided that $y > \sqrt{19\pi/40}$ and $\beta$ is sufficiently small.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

H.-S. Ding acknowledges support from NSFC (11461034), the NSF of Jiangxi Province (20143ACB21001), and the Foundation of Jiangxi Provincial Education Department (GJJ150342). M. M. Liu acknowledges support from the Graduate Innovation Fund of Jiangxi Province (YC2016-S157). The research of J. J. Nieto has been partially supported by AEI and the Ministerio de Economia y Competitividad of Spain under Grant MTM2016-75140-P and XUNTA de Galicia under Grant GRC2015-004 and cofinanced by the European Community fund FEDER.

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