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Characterizing Quantifier Fuzzification Mechanisms: a behavioral guide for applications

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Abstract

Important advances have been made in the fuzzy quantification field. Nevertheless, some problems remain when we face the decision of selecting the most convenient model for a specific application. In the literature, several desirable adequacy properties have been proposed, but theoretical limits impede quantification models from simultaneously fulfilling every adequacy property that has been defined. Besides, the complexity of model definitions and adequacy properties makes very difficult for real users to understand the particularities of the different models that have been presented. In this work we will present several criteria conceived to help in the process of selecting the most adequate Quantifier Fuzzification Mechanisms for specific practical applications. In addition, some of the best known well-behaved models will be compared against this list of criteria. Based on this analysis, some guidance to choose fuzzy quantification models for practical applications will be provided.

Keywords:
fuzzy quantification, determiner fuzzification schemes, theory of generalized quantifiers, quantifier fuzzification mechanism, applications of fuzzy quantification

1. Introduction

The evaluation of fuzzy quantified expressions is a topic that has been widely dealt with in literature [2, 7, 8, 10, 11, 12, 13, 14, 16, 17, 19, 20, 23, 29, 33, 35, 37, 39, 40]. The range of applications of fuzzy quantification includes fuzzy control [36], temporal reasoning in robotics [26], fuzzy databases [5], information retrieval [4, 13, 24, 25], data fusion [21, 37], syllogistic reasoning [27, 28] and more recently data-to-text applications [15, 30, 31, 32].

Moreover, the definition of adequate models to evaluate quantified expressions is fundamental to perform 'computing with words', topic that was suggested by Zadeh [41] to express the ability of programming systems in a linguistic way.

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In general, most approaches to fuzzy quantification use the concept of *fuzzy linguistic quantifier* to represent absolute or proportional fuzzy quantities. Zadeh [40] *defined quantifiers of the first type* as quantifiers used for representing absolute quantities (by using fuzzy numbers on $\mathbb{N}$) , and *quantifiers of the second type* as quantifiers used for representing relative quantities (defined by using fuzzy numbers on $[0, 1]$).

For analyzing the behavior of fuzzy quantification models different properties of convenient or necessary fulfillment have been defined [7, 9, 12, 20]. However, most of the approaches fail to exhibit a plausible behavior as it has been proved through the different reviews that have been published [2, 7, 8, 9, 20] and only a few [7, 11, 19, 20] seem to exhibit an adequate behavior in the general case.

In this work, we will follow Glöckner's approximation to fuzzy quantification [20]. In his approach, the author generalizes the concept of *generalized classic quantifier* [3] (second order predicates or set relationships) to the fuzzy case; that is, a *fuzzy quantifier* is a fuzzy relationship between fuzzy sets. And then he rewrites the fuzzy quantification problem as the problem of looking for a mechanism to transform *semi-fuzzy quantifiers* (quantifiers in a middle point between generalized classic quantifiers and fuzzy quantifiers, used to specify the meaning of quantified expressions) into fuzzy quantifiers. The author calls these transformation mechanisms *Quantifier Fuzzification Mechanisms* (*QFMs*). Being based in the linguistic *Theory of Generalized Quantifiers (TGQ)* [3], this approach is able to handle most of the quantification phenomena of natural language. In addition, including quantification into a common theoretical framework following TGQ, it also allows the translation of most of the analysis that has been made from a linguistic perspective to the fuzzy case, and facilitates the definition and the test of adequacy properties.

Glöckner has also defined a rigorous axiomatic framework to ensure the good behavior of QFMs. Models fulfilling this framework are called *Determiner Fuzzification Schemes (DFSs)* and they comply with a broad set of properties that guarantee a good behavior from a linguistic and fuzzy point of view. See the recent [33] or [20] for a comparison between Zadeh's and Glöckner's approaches.

The DFS framework has supposed a notable advance and several well behaved QFMs have been identified [20], [9]. However, important problems still remain when we must face the decision of selecting an specific QFM for a practical application. First, it has been proved that no model can fulfill every desirable adequacy property that have been proposed [20], and as a consequence, a 'perfect model' cannot exist. Besides, the complexity of the definition of the models and adequacy properties makes really difficult for a user to decide which one is the most convenient for a certain application. In addition, as we will show along the exposition, there are some criteria that have not been previously taken into account for analyzing the plausible models and, even for the cases in which some criteria had been previously considered, a complete comparison among the behavior of at least the best-behaved models has not been done.

In this work we will focus in, to the best of our knowledge, the best behaved QFMs: models $\mathcal{F}^{MD}$, $\mathcal{F}^{I}$, $\mathcal{F}^{A}$ [9] and models $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$ [20] with the objective of establishing a set of criteria that facilitates the understanding of the behavioral differences among them and helps in the process of selecting the more convenient model for applications. All the selected models, being QFMs, present a more general definition than models following Zadeh's framework [40]. Furthermore, some of them generalize other known approaches, as the ones based on the Sugeno or Choquet integrals. Thus, selected models comprise a really good representation of the 'state of the art' of fuzzy quantification. We refer the reader to the exhaustive and recent revision in [8] for a thoroughly comparative analysis of fuzzy quantification proposals. Previous state of the art revisions about the fuzzy quantification field can be found in [2, 7, 9, 20].

Before continuing, we would remark that only the models $\mathcal{F}^A$, $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$ fulfill the strict DFS framework, being alpha-cut based models like $\mathcal{F}^{MD}$ and $\mathcal{F}^I$, previously considered as non plausible from the point of view of the DFS framework [20, section 7.2]. In order to understand the differences between these models and DFSs, we will first compare the selected models against the main properties considered into the QFM framework. Once the main differences derived from the properties described in [20] have been presented, we will introduce the new set of criteria that will allow us to improve the comparison between the different models and to prove that, for some problems, alpha-cut based models $\mathcal{F}^{MD}$ and $\mathcal{F}^I$ can be superior to known DFSs.

Moreover, as we will argue when we analyze the different models against the set of criteria introduced in this paper, 'a clear winner' cannot be identified, being the general situation that some models are more appropriate for some applications than others.

The paper is organized as follows. Section 2 will summarize Glöckner's approach to fuzzy quantification, based on QFMs. In section 3, we will present the definition of the models $\mathcal{F}^{MD}$, $\mathcal{F}^I$, $\mathcal{F}^A$, $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$. Section 4 will present the main properties considered in the QFM framework [20] and a brief comparison of the models $\mathcal{F}^{MD}$, $\mathcal{F}^I$, $\mathcal{F}^A$, $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$, with the objective of clearly identifying the behavioral differences of these models with respect to these properties. Section 5 will be devoted to establish the set of criteria that will allow us to improve the comparison of the considered models, and to analyze the different models against this new set of criteria. Section 6 summarizes the results and establishes some criteria to guide in the model selection for applications. The paper is closed with some conclusions.

2. The fuzzy quantification framework

To overcome Zadeh's framework to fuzzy quantification Glöckner [20] rewrote the problem of fuzzy quantification as the problem of looking for adequate ways to convert specification means (semi-fuzzy quantifiers) into operational means (fuzzy quantifiers).

Fuzzy quantifiers are just a fuzzy generalization of crisp or classic quantifiers. Before giving the definition of fuzzy quantifiers, we will show the definition of classic quantifiers according to TGQ.

Definition 1. *A two valued (generalized) quantifier on a base set $E \neq \varnothing$ is a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$, where $n \in \mathbb{N}$ is the arity (number of arguments) of Q, $\mathbf{2} = \{0, 1\}$ denotes the set of crisp truth values, and $\mathcal{P}(E)$ is the powerset of E.*

Examples of some definitions of classic quantifiers are:

$$\mathbf{all}\,(Y_1, Y_2) = Y_1 \subseteq Y_2$$

$$\mathbf{at\_least}80\%\,(Y_1, Y_2) = \begin{cases} \frac{|Y_1 \cap Y_2|}{|Y_1|} \geq 0.80 & Y_1 \neq \varnothing \\ 1 & Y_1 = \varnothing \end{cases}$$

In a fuzzy quantifier, arguments and results can be fuzzy. A fuzzy quantifier assigns a gradual result to each choice of $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, where by $\widetilde{\mathcal{P}}(E)$ we denote the fuzzy powerset of E. In the case E is finite we will denote $|E| = m$.

Definition 2. *[20, definition 2.6] An n-ary fuzzy quantifier $\widetilde{Q}$ on a base set $E \neq \varnothing$ is a mapping $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I} = [0, 1]$.*

For example, the fuzzy quantifier $\widetilde{\mathbf{all}} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ could be defined as:

$$\widetilde{\mathbf{all}}(X_1, X_2) = \inf\{\max(1 - \mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\}$$

where by $\mu_X(e)$ we denote the membership function of $X \in \widetilde{\mathcal{P}}(E)$ and by inf we are denoting the infimum.

Although a certain consensus may be achieved to accept the previous expression as a suitable definition for $\widetilde{\mathbf{all}}$ this has not to be the unique possible one. The problem of establishing consistent fuzzy definitions for quantifiers (e.g., *'at least eighty percent'*) is faced in [20] by introducing the concept of semi-fuzzy quantifiers. A semi-fuzzy quantifier represents a medium point between classic quantifiers and fuzzy quantifiers. Semi-fuzzy quantifiers are similar but far more general than Zadeh's linguistic quantifiers [40]. A semi-fuzzy quantifier only accepts crisp arguments, as classic quantifiers, but let the result range over the truth grade scale $\mathbf{I}$, as for fuzzy quantifiers.

Definition 3. *[20, definition 2.8] An n-ary semi-fuzzy quantifier Q on a base set $E \neq \varnothing$ is a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

Q assigns a gradual result to each pair of crisp sets $(Y_1, \ldots, Y_n)$. Examples of semi-fuzzy quantifiers are:

$$\mathbf{about\_5}(Y_1, Y_2) = T_{2,4,6,8}(|Y_1 \cap Y_2|) \tag{1}$$

$$\mathbf{at\_least\_about}80\%(Y_1, Y_2) = \begin{cases} S_{0.5,0.8}\left(\frac{|Y_1 \cap Y_2|}{|Y_1|}\right) & X_1 \neq \varnothing \\ 1 & X_1 = \varnothing \end{cases}$$

where $T_{2,4,6,8}(x)$ and $S_{0.5,0.8}(x)$ represent the common trapezoidal and S fuzzy numbers[1].

Semi-fuzzy quantifiers are much more intuitive and easier to define than fuzzy quantifiers, but they do not solve the problem of evaluating fuzzy quantified sentences. In fact, additional mechanisms are needed to transform semi-fuzzy quantifiers into fuzzy quantifiers, i.e., mappings with domain in the universe of semi-fuzzy quantifiers and range in the universe of fuzzy quantifiers:

Definition 4. *[20, definition 2.10]A quantifier fuzzification mechanism (QFM) $\mathcal{F}$ assigns to each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \rightarrow \mathbf{I}$ a corresponding fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \rightarrow \mathbf{I}$ of the same arity $n \in \mathbb{N}$ and on the same base set E.*

3. Some paradigmatic QFMs

3.1. Standard DFSs based on trivalued cuts

In this section we will present the three main Glöckner's approaches [20]. All the models that have been proposed by Glöckner are standard DFSs, and as a consequence they show an excellent

[1]Functions $T_{a,b,c,d}$ and $S_{\alpha,\gamma}$ are defined as

$$T_{a,b,c,d}(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & b < x \leq c \\ 1 - \frac{x-c}{d-c} & c < x \leq d \\ 0 & d < x \end{cases}, S_{\alpha,\gamma}(x) = \begin{cases} 0 & x < \alpha \\ 2\left(\frac{(x-\alpha)}{(\gamma-\alpha)}\right)^2 & \alpha < x \leq \frac{\alpha+\gamma}{2} \\ 1 - 2\left(\frac{(x-\gamma)}{(\gamma-\alpha)}\right)^2 & \frac{\alpha+\gamma}{2} < x \leq \gamma \\ 1 & \gamma < x \end{cases}$$

theoretical behavior. These models are called standard because they induce the standard *tnorm min* and the standard *tconorm max*.[2]

Before presenting models $\mathcal{M}$ [20, definition 7.22], $\mathcal{M}_{CX}$ [20, definition 7.56] and $\mathcal{F}_{owa}$ [20, definition 8.13] we need to introduce some additional definitions.

Definition 5. *[20, definition 7.15]Let E be a referential set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy set, and $\gamma \in \mathbf{I}$. $X_\gamma^{\min}, X_\gamma^{\max} \in \mathcal{P}(E)$ are defined as:*

$$X_\gamma^{\min} = \begin{cases} X_{>\frac{1}{2}} & : \quad \gamma = 0 \\ X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} & : \quad \gamma > 0 \end{cases}$$
$$X_\gamma^{\max} = \begin{cases} X_{\geq\frac{1}{2}} & : \quad \gamma = 0 \\ X_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \quad \gamma > 0 \end{cases}$$

where $X_{\geq\alpha} = \{e \in E : \mu_X(e) \geq \alpha\}$ is the alpha-cut of level α of X and $X_{>\alpha} = \{e \in E : \mu_X(e) > \alpha\}$ is the strict alpha-cut of level α.

In previous expression, $X_\gamma^{\min}$ represents the elements that without doubt, belong to the fuzzy set X for the 'cautiousness' level γ whilst $X_\gamma^{\max}$ includes also the elements whose belongingness to the cautiousness level γ is undefined. Elements that are not in $X_\gamma^{\max}$ do not belong to the cautiousness level γ. The cautiousness cut can be interpreted as a 'trivalued set' in which elements in $X_\gamma^{\min}$ have membership function of 1, whilst for elements in $X_\gamma^{\max}/X_\gamma^{\min}$ belongingness is undefined (membership degree of $\frac{1}{2}$). Membership function for elements that are not in $X_\gamma^{\max}$ is 0.

For the definition of $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$ we also need the fuzzy median operator:

Definition 6. *Fuzzy median $med_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ is defined as:*

$$med_{\frac{1}{2}}(u_1, u_2) = \begin{cases} \min(u_1, u_2) & : \quad \min(u_1, u_2) > \frac{1}{2} \\ \max(u_1, u_2) & : \quad \max(u_1, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{otherwise} \end{cases}$$

The following definitions extends the fuzzy median to fuzzy sets:

Definition 7. *Operator $m_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \to \mathbf{I}$ is defined as*

$$m_{\frac{1}{2}} X = med_{\frac{1}{2}}(\inf X, \sup X)$$

for all $X \in \mathcal{P}(\mathbf{I})$, where by $\inf X, \sup X$ we are denoting the infimum and supremum of X, respectively.

The set that contains all the possible images of a quantifier over the range defined by a three valued cut of level γ is defined as [19, page 100]:

Definition 8. *Let $Q : \mathcal{P}(E) \to \mathbf{I}$ be a semi-fuzzy quantifier, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ fuzzy sets and $\gamma \in [0,1]$ a cautiousness level. $S_{Q,X_1,\ldots,X_n}(\gamma) : [0,1] \to \mathbf{I}$ is defined as:*

$$S_{Q,X_1,\ldots,X_n}(\gamma)(X_1, \ldots, X_n) = \left\{Q(Y_1, \ldots, Y_n) : (X_i)_\gamma^{\min} \subseteq Y_i \subseteq (X_i)_\gamma^{\max}\right\}$$

[2]In [20, section 3.4] it is explained how semi-fuzzy quantifiers can be used 'to embed' the classical logical functions. By means of the application of a QFM $\mathcal{F}$, we can study if $\mathcal{F}$ transforms the classical logical functions into appropriate fuzzy logical functions.

Supremum and infimum of $S_{Q,X_1,\ldots,X_n}(\gamma)$ are represented by means of the following notation:

Definition 9. *Let $Q : \widetilde{\mathcal{P}}(E) \to \mathbf{I}$ be a semi-fuzzy quantifier, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ fuzzy sets and $\gamma \in [0,1]$ a cautiousness level. $\top_{Q,X_1,\ldots,X_n}(\gamma) : [0,1] \to \mathbf{I}$ is defined as:*

$$\top_{Q,X_1,\ldots,X_n}(\gamma) = \sup S_{Q,X_1,\ldots,X_n}(\gamma)$$

Definition 10. *Let $Q : \widetilde{\mathcal{P}}(E) \to \mathbf{I}$ be a semi-fuzzy quantifier, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ fuzzy sets and $\gamma \in [0,1]$ a cautiousness level. $\perp_{Q,X_1,\ldots,X_n}(\gamma) : [0,1] \to \mathbf{I}$ is defined as:*

$$\perp_{Q,X_1,\ldots,X_n}(\gamma) = \inf S_{Q,X_1,\ldots,X_n}(\gamma)$$

Using previous definitions we present the three paradigmatic DFSs :

Definition 11. *[20, definition 7.22] Standard DFS $\mathcal{M} : (Q : \mathcal{P}(E) \to \mathbf{I}) \to (\widetilde{Q} : \widetilde{\mathcal{P}}(E) \to \mathbf{I})$ is defined as*

$$\mathcal{M}(Q)(X_1, \ldots, X_n) = \int_0^1 med_{\frac{1}{2}}\left(\top_{Q,X_1,\ldots,X_n}(\gamma), \perp_{Q,X_1,\ldots,X_n}(\gamma)\right) d\gamma$$

Definition 12. *[20, definition 7.56, theorem 7.87] Standard DFS $\mathcal{M}_{CX} : (Q : \mathcal{P}(E) \to \mathbf{I}) \to \left(\widetilde{Q} : \widetilde{\mathcal{P}}(E) \to \mathbf{I}\right)$ is defined as*

$$\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) = \sup\left\{Q^L_{V,W}(X_1, \ldots, X_n) : V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\right\}$$

where

$$Q^L_{V,W}(X_1, \ldots, X_n) = \min\left(\Xi_{V,W}(X_1, \ldots, X_n), \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i\}\right)$$
$$\Xi_{V,W}(X_1, \ldots, X_n) = \min_{i=1}^{n} \min\left(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\}\right)$$

Definition 13. *[20, definition 8.13] Standard DFS $\mathcal{F}_{owa} : (Q : \mathcal{P}(E) \to \mathbf{I}) \to \left(\widetilde{Q} : \widetilde{\mathcal{P}}(E) \to \mathbf{I}\right)$ is defined as*

$$\mathcal{F}_{owa}(Q)(X_1, \ldots, X_n) = \frac{1}{2}\int_0^1 \top_{Q,X_1,\ldots,X_n}(\gamma)\, d\gamma + \frac{1}{2}\int_0^1 \perp_{Q,X_1,\ldots,X_n}(\gamma)\, d\gamma$$

3.2. Alpha-cut based QFMs $\mathcal{F}^I$ and $\mathcal{F}^{MD}$

Now, we will present the two QFMs based on alpha-cuts $\mathcal{F}^I$ and $\mathcal{F}^{MD}$.

Definition 14. *[11, section 2.1], [9, chapter 3]Let $Q : \mathcal{P}(E)^n \to \mathbf{I}$ be a semi-fuzzy quantifier over a base set E. The QFM $\mathcal{F}^{MD}$ is defined as:*

$$\mathcal{F}^{MD}(Q)(X_1, \ldots, X_n) = \int_0^1 Q\left((X_1)_{\geq\alpha}, \ldots, (X_n)_{\geq\alpha}\right) d\alpha$$

for every $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

When fuzzy sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are normalized and we limit ourselves to the unary and binary quantifiers considered in the Zadeh's framework, $\mathcal{F}^{MD}$ coincides with the quantification model *GD* defined in [6, page 281], [34, section 3.3.2. and section 3.4.1.], [7, page 37]. In this way, $\mathcal{F}^{MD}$ generalizes the *GD* model to the Glöckner's framework.

Let us now present the definition of the $\mathcal{F}^I$ model.

Definition 15. *[10], [11, section 2.2],[9, chapter 3] Let* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *be a semi-fuzzy quantifier over a base set E. The QFM* $\mathcal{F}^I$ *is defined as:*

$$\mathcal{F}^I(Q)(X_1, \ldots, X_n) = \int_0^1 \ldots \int_0^1 Q\left((X_1)_{\geq \alpha_1}, \ldots, (X_n)_{\geq \alpha_n}\right) d\alpha_1 \ldots d\alpha_n$$

for every $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

3.3. Non standard DFS $\mathcal{F}^A$

The definition of the QFM $\mathcal{F}^A$ is based on a probabilistic interpretation of fuzzy sets in which we interpret membership degrees as probabilities [9],[12]. However, the $\mathcal{F}^A$ model can also be defined by means of fuzzy operators without any reference to probability theory.

The QFM $\mathcal{F}^A$ fulfills the axioms of the DFS framework but it is not a standard DFS, as the logic operators induced by the $\mathcal{F}^A$ model are the product *tnorm* and the probabilistic sum *tconorm*.

Definition 16. *Let* $X \in \widetilde{\mathcal{P}}(E)$ *be a fuzzy set, E finite. The probability of the crisp set* $Y \in \mathcal{P}(E)$ *of being a representative of the fuzzy set* $X \in \widetilde{\mathcal{P}}(E)$ *is defined as*

$$m_X(Y) = \prod_{e \in Y} \mu_X(e) \prod_{e \in E \setminus Y} (1 - \mu_X(e))$$

As we have stated above, it is possible to make a similar definition without making any reference to probability theory. If we consider the product *tnorm* ($\wedge(x_1, x_2) = x_1 \cdot x_2$) and the Lukasiewicz implication then $m_X(Y)$ is the *equipotence* between Y and X [1]:

$$Eq(Y, X) = \wedge_{e \in E} (\mu_X(e) \to \mu_Y(e)) \wedge (\mu_Y(e) \to \mu_X(e))$$

Using the previous definition we define the $\mathcal{F}^A$ DFS as:

Definition 17. *[14, pag. 1359]Let* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *be a semi-fuzzy quantifier, E finite. The DFS* $\mathcal{F}^A$ *is defined as*

$$\mathcal{F}^A(Q)(X_1, \ldots, X_n) = \sum_{Y_1 \in \mathcal{P}(E)} \ldots \sum_{Y_n \in \mathcal{P}(E)} m_{X_1}(Y_1) \ldots m_{X_n}(Y_n)\, Q(Y_1, \ldots, Y_n) \tag{2}$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

The next expression is an alternative definition of the model $\mathcal{F}^A$:

$$\mathcal{F}^A(X_1, \ldots, X_n) = \bigvee_{Y_1 \in \mathcal{P}(E)} \ldots \bigvee_{Y_n \in \mathcal{P}(E)} Eq(Y_1, X_1) \wedge \ldots \wedge Eq(Y_n, X_n) \wedge Q(Y_1, \ldots, Y_n)$$

where $\vee$ is the Lukasiewicz *tconorm* ($\vee(x_1, x_2) = \min(x_1 + x_2, 1)$), $\wedge$ is the product *tnorm* ($\wedge(x_1, x_2) = x_1 \cdot x_2$) and $Eq(Y, X)$ is the equipotence between the crisp set Y and the fuzzy set X. In this way, the definition of $\mathcal{F}^A$ can be done by means of common fuzzy operators.

4. The DFS axiomatic framework

In this section we will present the DFS axiomatic framework [20]. In the previous reference, the author has dedicated the whole third and fourth chapters to describe the framework and the properties that are a consequence of it. For the sake of brevity, we will only give a general overview of the framework and some intuitions about the set of properties derived from it. We refer the reader to the previous publication for full detail.

Definition 18. *A QFM $\mathcal{F}$ is called a determiner fuzzification scheme (DFS) if the following conditions are satisfied for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \to \mathbf{I}$:*

Correct generalization	$\mathcal{U}(\mathcal{F}(Q)) = Q \quad \text{if } n \leq 1$	(Z-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e \quad \text{if } Q = \pi_e$ for some $e \in E$	(Z-2)
Dualisation	$\mathcal{F}\left(Q\widetilde{\square}\right) = \mathcal{F}(Q)\widetilde{\square} \quad n > 0$	(Z-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup} \quad n > 0$	(Z-4)
Preservation of monotonicity	If Q is nonincreasing in the n-th arg, then	(Z-5)
	$\mathcal{F}(Q)$ is nonincreasing in n-th arg, $n > 0$	
Functional application	$\mathcal{F}\left(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f_i}\right) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$	(Z-6)
	where $f_1, \ldots, f_n : E' \to E, E' \neq \varnothing$	

4.1. Main properties derived from the DFS framework

We will only make a brief exposition of the main properties derived from the DFS framework. Full detail can be found in the aforementioned reference [20, chapters three and four.].

- **Correct generalization (P1):** perhaps, the most important property derived from the DFS framework is the *correct generalization* property. Correct generalization requires that the behavior of a fuzzy quantifier $\mathcal{F}(Q)$ over crisp arguments is the expected; that is, results obtained with a fuzzy quantifier $\mathcal{F}(Q)$ and with the corresponding semi-fuzzy quantifier Q must coincide over crisp arguments. It is included in the DFS axiomatic framework for semi-fuzzy quantifiers of arities 0 and 1 (Z-1).

- **Quantitativity (P2):** quantitative quantifiers do not depend on any particular characteristic of the elements of the base set. In the finite case, quantitative quantifiers can always be defined as a function of the cardinality of the boolean combinations of the argument sets. A QFM $\mathcal{F}$ preserves quantitativity if quantitative semi-fuzzy quantifiers are translated into quantitative fuzzy quantifiers by $\mathcal{F}$. Most typical examples of quantifiers we find in the literature are quantitative (e.g., *'around five'*, *'at least 80%'*, etc.). Non-quantitative quantifiers involve the reference to specific elements of the referential (e.g., *'John'* in a set of people).

- **Projection quantifier (P3):** Axiom Z-2 guarantees that the *projection crisp quantifier* $\pi_e(Y)$ (that returns 1 if $e \in Y$ and 0 in other case) is generalized to the *fuzzy projection quantifier* $\widetilde{\pi}_e(X)$ (that returns $\mu_X(e)$).

- **Induced propositional logic (P4):** we will say that a QFM comply with the induced propositional logic if crisp logical functions ($\neg(x)$, $\wedge(x_1, x_2)$, $\vee(x_1, x_2)$, $\to(x_1, x_2)$), that can be embedded into the definition of semi-fuzzy quantifiers, are generalized to acceptable fuzzy logical functions; that is, a negation operator, a *tnorm*, a *tconorm* and a fuzzy implication function. For example, the negation function can be embedded in the following semi-fuzzy quantifier $Q_{\neg} : \mathcal{P}(\{e\}) \to \{0,1\}$ such that $Q_{\neg}(\{e\}) = 0$ and $Q(\varnothing) = 1$. We should expect that $\mathcal{F}(Q_{\neg})$ would be a fuzzy negation operator.

- **External negation (P5):** in the common case, external negation of a semi-fuzzy quantifier is computed by the application of the standard negation $\widetilde{\neg}(x) = 1 - x$. A QFM fulfilling the external negation property guarantees that $\mathcal{F}(\widetilde{\neg}Q)$ is equivalent to $\widetilde{\neg}\mathcal{F}(Q)$. Thanks to the external negation property, equivalence of expressions *"it is false that at least 80% of the hard workers are well paid"* and *"less than 80% of the hard workers are well paid"* is assured.

- **Internal negation (P6):** the internal negation (antonym) of a semi-fuzzy quantifier is defined as $Q\neg(Y_1,\ldots,Y_n) = Q(Y_1,\ldots,\neg Y_n)$. For example, *'no'* is the antonym of *'all'* because $\mathbf{all}\neg(Y_1,Y_2) = \mathbf{all}(Y_1,\neg Y_2) = \mathbf{no}(Y_1,Y_2)$. Fulfillment of the internal negation property assures that internal negation transformations are translated to the fuzzy case; that is, we guarantee that $\mathcal{F}(\widetilde{\mathbf{all}\neg})(X_1,X_2) = \mathcal{F}(\mathbf{all})(X_1,\widetilde{\neg}X_2) = \mathcal{F}(\mathbf{no})(X_1,X_2)$ where $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ are fuzzy sets. In combination with external negation property, it assures 'duality transformations' (see below).

- **Dualisation (P7):** the dualisation property coincides with the Z-3 axiom of the DFS framework, being a consequence of the simultaneous fulfillment of the external negation and internal negation properties. The dual of a semi-fuzzy quantifier Q is defined as $Q\widetilde{\square}(Y_1,\ldots,Y_n) = \widetilde{\neg}Q\neg(Y_1,\ldots,Y_n) = \widetilde{\neg}Q(Y_1,\ldots,\neg Y_n)$ where $Y_1,\ldots,Y_n \in \mathcal{P}(E)$ are crips sets, whilst the corresponding definiton for fuzzy quantifiers is $\widetilde{Q\square}(X_1,\ldots,X_n) = \widetilde{\neg}\widetilde{Q}\widetilde{\neg}(X_1,\ldots,X_n) = \widetilde{\neg}\widetilde{Q}(X_1,\ldots,\widetilde{\neg}X_n)$ where $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$ are fuzzy sets. In conjunction with previous properties, equivalences in the 'Aristotelian square' are maintained in the fuzzy case. As an example, equivalence of $\mathcal{F}(\mathbf{all})(\mathbf{hard\_workers}, \mathbf{well\_paid})$ and $\mathcal{F}(\mathbf{no})(\mathbf{hard\_workers}, \widetilde{\neg}\mathbf{well\_paid})$ is assured, or in words, *"all hard workers are well paid"* is equivalent to *"no hard worker is not well paid"*.

- **Union/intersection of arguments (P8)**: this property guarantees the compliance with some transformations that allow to construct new quantifiers by means of unions (and intersections) of arguments. As a particular case, the equivalence between absolute unary and binary quantifiers is a consequence of this axiom. As an example, the equivalence between *"about 5 hard workers are well paid"* and *"about 5 people are hard workers and well paid"* is assured. For QFMs fulfilling the DFS framework, the fulfillment of this property, in combination with internal and external negation properties, allow the preservation of the boolean argument structure that can be expressed in natural language when none of the boolean variables X_i occurs more than once [20, section 3.6].

- **Coherence with standard quantifiers (P9):** by standard quantifiers we refer to the classical quantifiers $\exists, \forall$ and their binary versions **some** and **all**. We will say that a QFM maintains coherence with standard quantifiers if the fuzzy versions of the classical quantifiers are the expected. For example, a QFM fulfilling this property complies (where $\widetilde{\vee}, \widetilde{\wedge}, \widetilde{\to}$ are the logical operators induced by the QFM):

$$\mathcal{F}(\exists)(X) = \sup\left\{\mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E), a_i \neq a_j \text{ if } i \neq j\right\}$$

$$\mathcal{F}(\mathbf{all})(X_1, X_2) = \inf\left\{\mathop{\widetilde{\wedge}}_{i=1}^{m} \mu_{X_1}(a_i) \mathbin{\widetilde{\rightarrow}} \mu_{X_2}(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E), a_i \neq a_j \text{ if } i \neq j\right\}$$

- **Monotonicity in arguments (P10):** this property assures the translation of monotonicity in arguments relations from the semi-fuzzy to the fuzzy case. As an example, the binary semi-fuzzy quantifier *'most'* is increasing in its second argument (e.g. *"most students are poor"*). This property assures that the fuzzy version of *'most'* is also increasing in its second argument.

- **Monotonicity in quantifiers (P11):** this property assures the preservation of monotonicity relations in quantifiers. For example, *'between 4 and 6'* is more specific than *'between 2 and 8'*. Fulfilment of this property assures that in the fuzzy case, the specificity relations between quantifiers are preserved.

- **Crisp argument insertion (P12):** For a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \to \mathbf{I}$, crisp argument insertion allows to construct a new quantifier $Q : \mathcal{P}(E)^{n-1} \to \mathbf{I}$ by means of the restriction of Q by a crisp set A; that is, the crisp argument insertion $Q \lhd A$ is defined as $Q \lhd A(Y_1, \ldots, Y_{n-1}) = Q(Y_1, \ldots, Y_{n-1}, A)$. A QFM preserving the property of crisp argument insertion assures that $\mathcal{F}(Q \lhd A) = \mathcal{F}(Q) \lhd A$; that is, it is equivalent to first restrict the semi-fuzzy quantifier Q by A and then applying the fuzzification scheme $\mathcal{F}$ or to first applying the fuzzification mechanism and then restricting the corresponding fuzzy quantifier by A. Crisp argument insertion allows to model the 'adjectival restriction' of natural language in the crisp case.

4.2. Some relevant properties non included in the DFS framework

In [20, chapter six] some additional adequacy properties for characterizing DFSs were described. These additional properties were not included in the DFS framework in some cases, for being incompatible with it, and in other cases, in order to not excessively restrict the set of theoretical models fulfilling the framework, which was important for the author for studying the full set of classes of standard models and their theoretical limits. We will present now the more relevant:

- **Continuity in arguments (P13):** this property assures the continuity of the models with respect to the argument sets. It is fundamental to guarantee that small modifications in arguments do not provoke high variations in the results of evaluating quantified expressions.

- **Continuity in quantifiers (P14):** this property assures the continuity of the models with respect to variations in the quantifiers.

- **Propagation of fuzziness (P15):** this property assures that fuzzier inputs (understood as fuzzier input sets) and fuzzier quantifiers produce fuzzier outputs. We will discuss this property in more detail when we introduce the set of criteria we will use to improve the characterization of the behavior of the QFMs (see section 5.3).

- **Fuzzy argument insertion (P16):** this property is the fuzzy counterpart of the crisp argument insertion. It is a very restrictive property, that will impose great limitations into the set of models fulfilling the DFS axiomatic framework .

4.3. Comparison of the models against the QFM properties

In this section we will make a brief summary of the theoretical analysis of the QFMs $\mathcal{F}^{MD}$, $\mathcal{F}^{I}$, the non-standard DFS $\mathcal{F}^{A}$ and the standard DFSs $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$ with respect to the set of previous properties. Table 1 summarizes the fulfillment of the properties for the different models. A detailed analysis of these models can be found in [20] and [9].

This analysis will allow us to understand the main differences between the models we are considering. As we will argue in the following section, although the set of properties previously presented allows for a deep analysis of the models, we consider that they are not enough to understand the behavioral differences between them and to decide which ones can be more appropriate for specific applications. The introduction of these new criteria and the analysis of the behavior of the models with respect to it will be the objective of the last two sections of this paper.

Before summarizing the behavior of these models against these properties, we would like to emphasize that our point of view is that although models $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ are not DFSs, they are really competitive with respect to models fulfilling the DFS framework. The main differences between the $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ models when we compare them with DFSs is that they fail to fulfill some of the linguistic properties derived from the DFS framework. In addition, these models also fail to fulfill some of the QFM properties in the infinite case, which do not affect to most of the practical applications of fuzzy quantification[3]. Finally $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ only fulfill the coherence with standard quantifiers property in the unary case, although in the specific case of the $\mathcal{F}^{I}$ model fulfillment of the property depends on the mechanism we will use to compute the induced operators of the model [9, chapter 3].

The competitiveness of $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ models with respect to DFSs will become more clear when we will present the analysis of the models against the new set of criteria, which from our point of view will prove that in some cases models $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ present some advantages against models fulfilling the DFS framework.

4.3.1. $\mathcal{M}$ model

$\mathcal{M}$ model is one of the first models formulated by Glöckner [18] and it is also one of the three models for which the author has provided computational algorithms in [20, chapter 11]. Being a standard DFS $\mathcal{M}$ model fulfills the properties derived from the DFS framework. Additionally, the $\mathcal{M}$ model is *continuous in arguments* and *in quantifiers* and fulfills the properties of *propagation of fuzziness in arguments* and *in quantifiers*.

4.3.2. $\mathcal{M}_{CX}$ model

$\mathcal{M}_{CX}$ model is also an standard DFS and another model for which the author has provided a computational implementation in [20, chapter 11]. $\mathcal{M}_{CX}$ is *continuous in arguments* and *in quantifiers* and fulfills both *fuzziness propagation properties*. $\mathcal{M}_{CX}$ is considered by Glöckner

[3]We have the hypothesis that for 'practical quantifiers' (i.e., defined by means of continuous fuzzy numbers) models $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ fulfill the continuous in arguments property. We also have the hypothesis that model $\mathcal{F}^{I}$ fulfills the internal negation property for infinite domains in the same cases. The fulfillment of these properties will guarantee the convenience of these models for infinite domains in the practical cases.

	$\mathcal{M}$	$\mathcal{M}_{CX}$	$\mathcal{F}_{owa}$	$\mathcal{F}^{MD}$	$\mathcal{F}^{I}$	$\mathcal{F}^{A}$
Properties derived from the DFS framework						
P1. Correct Generalization	Y	Y	Y	Y	Y	Y
P2. Quantitativity	Y	Y	Y	Y	Y	Y
P3. Projection quantifiers	Y	Y	Y	Y	Y	Y
P4. Induced propositional logic	Y	Y	Y	Y	Y	Y
P6. External negation	Y	Y	Y	Y	Y	Y
P7. Internal negation	Y	Y	Y	N	finite	Y
P8. Dualisation	Y	Y	Y	N	finite	Y
P9. Union/intersection of argument	Y	Y	Y	Y	N	Y
P10. Coherence with standard quantifiers	Y	Y	Y	unary	unary	Y
P11. Monotonicity in arguments	Y	Y	Y	Y	Y	Y
P12. Monotonicity in quantifiers	Y	Y	Y	Y	Y	Y
P13. Crisp Argument Insertion	Y	Y	Y	Y	Y	Y
Additional properties						
P14. Continuity in arguments	Y	Y	Y	finite	finite	finite
P15. Continuity in quantifiers	Y	Y	Y	Y	Y	Y
P16. Propagation of fuzziness	Y	Y	N	N	N	N
P17. Fuzzy argument insertion	N	Y	N	N	Y	Y

Table 1: Comparison of the behavior of the models against the set of properties in the QFM framework.

as a model of unique properties: it fulfills the property of *fuzzy argument insertion* [20, definition 7.82], it is specially robust against modification of membership degrees and generalizes the Sugeno integral (see [20, section 7.13] for more details).

4.3.3. $\mathcal{F}_{owa}$ model

$\mathcal{F}_{owa}$ model is the paradigmatic example of an standard DFS that does not propagate *fuzziness in arguments* or *in quantifiers*. $\mathcal{F}_{owa}$ model is also *continuous in arguments* and *in quantifiers*. As it fails to fulfill propagation of fuzziness properties, it is considered as the ideal model for applications in which an improved discriminative power is necessary [20, section 8.1]. $\mathcal{F}_{owa}$ model generalizes Choquet integral. It is the third model for which a computational implementation has been provided in [20, definition 7.82].

4.3.4. $\mathcal{F}^{MD}$ model

$\mathcal{F}^{MD}$ is the generalization to *QFMs* of the *GD* model proposed by Delgado et al. in [6], [34], [7]. $\mathcal{F}^{MD}$ model is not a DFS, failing to fulfill the *internal negation property*, and as a consequence, the *dualisation axiom of DFSs (Z3)*. $\mathcal{F}^{MD}$ model is *continuous in the arguments in the finite case* and also *continuous in the quantifiers*. $\mathcal{F}^{MD}$ fulfills the properties of *probabilistic interpretation of quantifiers* and of *averaging for the identity quantifier* [9, chapter 3], that will be reintroduced as one of the criteria for comparing the behavior of selected QFMs in the following section. $\mathcal{F}^{MD}$ does not fulfill any of the *propagation of fuzziness properties*.

4.3.5. $\mathcal{F}^{I}$ model

$\mathcal{F}^{I}$ model is the second alpha-cut based model analyzed in [9]. $\mathcal{F}^{I}$ model does not fulfill the *internal joins property (axiom Z4)*, and then fails to be a DFS. $\mathcal{F}^{I}$ is *continuous in the arguments* *in finite domains* and also *continuous in the quantifiers*. $\mathcal{F}^I$ model fulfills the *dualisation property in the finite case*. $\mathcal{F}^I$ model also fulfills the properties of *probabilistic interpretation of quantifiers* and *averaging for the identity quantifier* [9, chapter 3]. $\mathcal{F}^I$ does not fulfill *propagation of fuzziness properties*.

4.3.6. $\mathcal{F}^A$ model

$\mathcal{F}^A$ model is, to our knowledge, the unique known non-standard DFS. The fuzzy operators induced by the model are the *product tnorm* and the *probabilistic sum tconorm*, making this model essentially different of the standard DFSs presented in [20]. By definition $\mathcal{F}^A$ is a finite model. Moreover, $\mathcal{F}^A$ is *continuous in arguments* and *in quantifiers*, it does not fulfill fuzziness propagation properties, but it fulfills the *probabilistic interpretation of quantifiers* and *averaging for the identity quantifier* properties.

5. Some additional criteria to characterize the behavioral differences of the QFMs

We have seen that models $\mathcal{F}^{MD}$, $\mathcal{F}^I$, $\mathcal{F}^A$, $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$ fulfill most of the adequacy properties that have been presented in [20]. If we only took into account properties included in the QFM framework when selecting a model for an application, we would just choose one of the best-behaved models (e.g., $\mathcal{F}^A$ or $\mathcal{M}_{CX}$) and we will use them in every possible application of fuzzy quantification.

However, as we will see through this section, properties included in the QFM framework are not enough to fully understand the behavioral differences between the selected models. We will present an analysis that proves that models here discussed have some strong differences in their behavior. In addition, an aspect we consider specially relevant is that, from an user viewpoint, the complexity of the definition of the models and adequacy properties make very difficult, for a non specialist in fuzzy quantification, to determine which model should be chosen for an specific application.

Thus, it is essential to establish a set of criteria that help us understand the behavioral differences between the models and facilitate the selection of the more convenient ones for applications. In general, the set of criteria that we will take into account would not allow us to select 'a perfect model', or even 'a preferred one' for every possible application. But we are convinced they are important to (1) clarify the differences between the behavior of the QFMs (2) to select or discard QFMs for specific applications with respect to the behavior we consider more important and (3) to understand the problems that the selection of a specific model could have for a particular application.

The following is a summary of the criteria we will consider:

- **Linguistic compatibility**. By linguistic compatibility we mean the fulfillment of the most relevant linguistic properties derived from the DFS framework. In the summary of the behavior of the main QFMs we have seen that between the selected models, only DFSs fulfill the main set of properties that have been established to guarantee an adequate behavior with respect to the main linguistic expectations.

- **Aggregative behavior for low degrees of membership:** aggregative behavior makes reference to the tendency of a model to confuse one 'high degree' membership element with a large quantity of 'low degree' membership elements. It has been one of the main critiques made to the $\sum count$ model [40].

- **Propagation of fuzziness:** Propagation of fuzziness [20, section 5.2 and 6.3] is the main property used to group the different classes of standard DFSs [20, chapters 7 and 8]. Basically, models fulfilling the propagation of fuzziness properties 'transfer' fuzziness in inputs to the outputs; that is, they guarantee that fuzzier inputs and/or fuzzier quantifiers produce fuzzier outputs.

- **Identity quantifier:** the 'identity semi-fuzzy quantifier' is defined by means of the identity function $f(x) = \frac{x}{m}$, $x \in 1, \ldots, m$ in the absolute case or by means of $f(x) = x, x \in [0, 1]$ in the proportional case. For this semi-fuzzy quantifier, a linear increase in the number of elements that belong to the input produces a linear increase in the output. We could expect that a reasonable fuzzy counterpart of the identity quantifier should also produce a linear increase in the output for a linear increase in the input.

- **Evaluating quantifiers over 'quantified partitions'.** With this criterion we refer us to the behavior of the models when we apply, simultaneously, a set of quantifiers dividing the quantification universe (e.g., *'nearly none'*, *'a few'*, *'several'*, *'many'*, *'nearly all'*) to a fuzzy set. That is, how the degrees of fulfillment of the evaluation of the quantified expressions are distributed between the labels.

- **Fine distinction between objects**. In applications of fuzzy quantification for ranking generation is generally needed that fuzzy quantifiers are able to clearly distinguish between objects fulfilling a set of criteria with different degrees. Criteria to distinguish the QFMs with respect to their discriminative power are necessary for these applications.

5.1. Linguistic compatibility

With linguistic compatibility we make reference to the main linguistic properties presented in [20, chapter 4 and 6]. The DFS framework guarantees that the main linguistic transformations, including *argument permutations, negation of quantifiers, antonym of quantifiers, dual of quantifiers, argument insertion, internal meets, etc.* are transferred from the semi-fuzzy to the fuzzy case.

5.1.1. Analysis of the models

Models $\mathcal{M}_{CX}$ and $\mathcal{F}^{A}$. Being DFSs, both models fulfill all the semantic linguistic properties derived from the DFS framework. Moreover, these models fulfill the fuzzy argument insertion property [20, section 6.8], as it can be seen in [20, section 7.13] and in [9, chapter 3].

Models $\mathcal{M}$ and $\mathcal{F}_{owa}$. $\mathcal{M}$ and $\mathcal{F}_{owa}$ models fulfill semantic linguistic properties derived from the DFS framework, but not fuzzy argument insertion.

Model $\mathcal{F}^{MD}$. The main difference of the $\mathcal{F}^{MD}$ model with respect to DFSs is the non-fulfillment of the *internal negation property*. This fact impedes the $\mathcal{F}^{MD}$ model to correctly translate antonym relationships to the fuzzy case and, as a consequence, duality transformations (see [9, chapter 3]). As an example, failing to fulfill the *internal negation property*, the model cannot guarantee the equivalence of $\mathcal{F}^{MD}$ (**all**) (**hard_workers**, **well_paid**) and $\mathcal{F}^{MD}$ (**no**) (**hard_workers**, $\widetilde{\neg}$**well_paid**). In words, results of evaluating *"all hard workers are well paid"* and *"no hard worker is not well paid"* are different.

$\mathcal{F}^{MD}$ fulfills the strong conservativity property [20, section 6.7] that guarantees that conservative semi-fuzzy quantifiers (i.e., quantifiers fulfilling $Q(Y_1, Y_2) = Q(Y_1, Y_1 \cap Y_2)$) are correctly translated to the fuzzy case [9, chapter 3]. This property is not fulfilled by any DFS. However, loosing the internal negation property and as a consequence, the maintenance of the relationships of the 'Aristotelian square' seems more relevant than the fulfillment of the conservativity property.

Model $\mathcal{F}^1$. Model $\mathcal{F}^1$ looses the *internal meets* property. Moreover, the internal negation property is only fulfilled in the finite case (see [9, chapter 3]).

Loosing the internal meets property, $\mathcal{F}^1$ model does not guarantee absolute unary/binary transformations. For example, $\mathcal{F}^1$ (about **_10**) (**hard_workers**, **well_paid**) and $\mathcal{F}^1$ (about **_10**) (**hard_workers**$\widetilde{\cap}$**well_paid**) are not equivalent, and then *"about 10 hard workers are well paid"* (evaluated by means of the binary absolute quantifier *"about 10"*) and *"about 10 employees are hard workers and are well paid"* (evaluated by means of the application of the unary version of the absolute quantifier *'about 10'* and the induced *tnorm* of the model used to compute the intersection of *'hard workers'* and *'well paid'*) will not produce the same results.

5.2. Aggregative behavior for low degrees of membership

Aggregative behavior for low degrees of membership is one of the main critiques that has been made to the Zadeh's $\sum count$ model [38],[20, section A.3], [2]. The intuition around aggregative behavior is that in evaluating quantified expressions, a large amount of elements fulfilling a property with 'low degree' of membership should not be confused with a small amount of elements fulfilling a property with 'high degree' of membership. In the case of the Zadeh's model is easy to understand the meaning of aggregative behavior as:

$$\sum count\,(\exists)\,(\{0.01/e_1,\ldots,0.01/e_{100}\}) = \sum count\,(\exists)\,(\{1/e_1, 0/e_2,\ldots,0/e_{100}\}) = 1$$

in words, *'exist one tall person'* can be fulfilled if there exists exactly *'one tall person'*, or if there exist 100 people being *'0.01 tall'*.

Although intuitions against aggregative behavior seem clear, giving up models presenting aggregative behavior will force us to discard non-standard DFS $\mathcal{F}$ associated to *archimedean tconorms*[4]. Being $\mathcal{F}(\exists)$ equal to [20, Theorem 4.61]:

$$\mathcal{F}(\exists)(X) = \sup\left\{\widetilde{\vee}_{i=1}^{m}(a_i) : A = \{a_1,\ldots,a_m\} \in \mathcal{P}(E)\ finite, a_i \neq a_j\ if\ i \neq j\right\}$$

for all $X \in \widetilde{\mathcal{P}}(E)$, then $\mathcal{F}(\exists)(X)$ will always present aggregative behavior for every non-standard DFS associated to an *archimedean tconorm* $\vee$. *Archimedean tconorms* are a very relevant class of *tconorm* operators, including most of the common examples of *tconorm* operators.

To the best of our knowledge, a clear definition of aggregative behavior has not been presented in the literature, that has limited itself to present examples with existential quantifiers and/or with proportional quantifiers representing small proportions (e.g., *'about 10%'*). In this discussion, we will limit us to consider aggregative behavior for existential quantifiers, as it is enough to characterize the models we are considering.

[4]For an *archimedean* tconorm, $\lim_{n\longrightarrow\infty} \vee(c/e_1,\ldots,c/e_n) = 1$. Every continuous tconorm such that $\vee(x,x) > x, x \in (0,1)$ is *archimedean*.

5.2.1. Analysis of the models

Model $\mathcal{F}^A$. $\mathcal{F}^A$ model presents aggregative behavior as a consequence of inducing the non-standard probabilistic sum *tconorm* $\widetilde{\vee}(a,b) = a + b - ab$. For the $\mathcal{F}^A$ model:

$$\mathcal{F}(\exists)(X) = \widetilde{\vee}_{e\in E}\mu_X(e)$$

Moreover, $\mathcal{F}^A$ model tends to the Zadeh's Sigma-count model when the size of the referential E tends to infinite [12]; that is:

$$\lim_{|E|\to\infty} \mathcal{F}^A(Q)(X) = \mu_Q\left(\frac{\sum_{e\in E}\mu_X(e)}{|E|}\right)$$

In this way, $\mathcal{F}^A$ shares the critiques of aggregative behavior that has been made to the Zadeh's model for large referential sets.

Models $\mathcal{M}$, $\mathcal{M}_{CX}$, $\mathcal{F}_{owa}$, $\mathcal{F}^{MD}$ and $\mathcal{F}^I$. None of the rest of the models show aggregative behavior. For all of them, $\mathcal{F}(\exists)(X) = \sup\{\mu_X(e) : e \in E\}$, E finite. We will give some intuitions about the reasons for which these models do not present aggregative behavior.

Model $\mathcal{M}_{CX}$ has been proved to be extremely stable. In [20, section 7.12] it is proved that a change in the arguments that does not exceed a given Δ will not change the result of the quantifier by more than Δ. Then, $\mathcal{F}(Q)(\varnothing)$ and $\mathcal{F}(Q)(\{c/e_1, c/e_2, \ldots, c/e_N\})$, with c 'small', will produce approximately the same results.

With respect to models $\mathcal{M}$, $\mathcal{F}_{owa}$, $\mathcal{F}^{MD}$ and $\mathcal{F}^I$ we should take into account that all of their definitions are made by using an integration process over the alpha-cuts or the three-valued cuts of the argument sets.

In the case of alpha cuts, only alpha cuts in the integration interval $(0, c]$ could be altered by modifications in degrees of membership of elements with membership degree $\mu_X(e) \leq c$ that are maintained in $(0, c]$. In the case of three-valued cuts, only the integration interval $[1 - 2c, 1]$ could be altered by modifications in degrees of membership in the same interval. As the results of the integral do not change out the integration range, effects of modifications lower or equal than c will be limited to c (in the case of alpha-cuts) or $2c$ in the case of three-valued cuts[5].

5.3. Propagation of fuzziness

Propagation of fuzziness is related with the transmission of imprecision from the inputs (arguments and quantifiers) to the outputs (results of evaluating quantified expressions). We will reproduce the main definitions in [20, section 5.2 and 6.3].

Let be $\preceq_c$ a partial order in $\mathbf{I} \times \mathbf{I}$ defined as

$$x \preceq_c y \Leftrightarrow y \leq x \leq \frac{1}{2} \text{ or } \frac{1}{2} \leq x \leq y$$

for $x, y \in \mathbf{I}$.

$\preceq_c$ can be extended to fuzzy sets, semi-fuzzy quantifiers and fuzzy quantifiers in the following way:

[5]It can be proved that differences in the integration process for $\mathcal{M}$ and $\mathcal{F}_{Ch}$ are also limited to c, but we are only interested in giving an intuitive explanation of the reasons for which these models do not present aggregative behaviour.

$$X \preceq_c X' \Leftrightarrow \mu_X(e) \preceq_c \mu_{X'}(e), \text{ for all } e \in E$$
$$Q \preceq_c Q' \Leftrightarrow Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n), \text{ for all } Y_1, \ldots, Y_n \in \mathcal{P}(E)$$
$$\widetilde{Q} \preceq_c \widetilde{Q'} \Leftrightarrow \widetilde{Q}(X_1, \ldots, X_n) \preceq_c \widetilde{Q'}(X_1, \ldots, X_n), \text{ for all } X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$$

Definition 19. *[20, section 6.3]. Let a QFM $\mathcal{F}$ be given.*

a. We say that $\mathcal{F}$ propagates fuzziness in arguments if the following property is satisfied for all $Q : \mathcal{P}(E)^n \to \mathbf{I}$ and $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$. If $X_i \preceq_c X'_i$ for all $i = 1, \ldots, n$ then $\mathcal{F}(Q)(X_1, \ldots, X_n) \preceq_c \mathcal{F}(Q)\left(X'_1, \ldots, X'_n\right)$.

b. We say that $\mathcal{F}$ propagates fuzziness in quantifiers if $\mathcal{F}(Q) \preceq_c \mathcal{F}(Q')$ whenever $Q \preceq_c Q'$.

Propagation of fuzziness in arguments and in quantifiers is considered as optional but really convenient in [20, section 6.3]. Intuitively, from an user point of view, fuzzier inputs or fuzzier quantifiers should not produce more specific outputs.

Although both propagation of fuzziness properties seem natural, we should note that most basic *tnorms* and *tconorms* do not fulfill propagation of fuzziness properties (e.g. product *tnorm* and Lukasiewicz *tnorm* and their corresponding *tconorms*) do not fulfill propagation of fuzziness. This fact is relevant, as every DFS embeds basic logic operators [20, section 3.4]. Moreover, fulfillment of propagation of fuzziness properties have strong negative consequences for the ranking of objects (see section 5.6).

5.3.1. Analysis of the models

Models $\mathcal{M}$, $\mathcal{M}_{CX}$. Models $\mathcal{M}$ and $\mathcal{M}_{CX}$ are the paradigmatic examples of standard DFSs fulfilling propagation of fuzziness properties (see [20, chapter 7]. Using $\mathcal{M}$ and $\mathcal{M}_{CX}$ assure that when presented with fuzzier inputs or quantifiers, we will always obtain fuzzier outputs.

Models $\mathcal{F}_{owa}, \mathcal{F}^A, \mathcal{F}^{MD}$ and $\mathcal{F}^I$. Model $\mathcal{F}_{owa}$ is the paradigmatic example of an standard DFSs that does not fulfill both propagation of fuzziness properties.

$\mathcal{F}^A$ model does not fulfill propagation of fuzziness in arguments, as it is not fulfilled by the induced product *tnorm* and the induced probabilistic sum *tconorm* of the model (see [9, chapter 3]) and it is easy to find counterexamples for propagation of fuzziness in quantifiers. $\mathcal{F}^{MD}$ and $\mathcal{F}^I$ do not fulfill the property of propagation of fuzziness in arguments (see [9, chapter 3]) and it is also trivial to find counterexamples for the property of propagation of fuzziness in quantifiers.

5.4. Identity quantifier: as many as possible

First, we will define the **identity** semi-fuzzy quantifier. We will limit us to the proportional case:

Definition 20. *The unary semi-fuzzy quantifier* ***identity****: $\mathcal{P}(E) \to \mathbf{I}$ is defined as*

$$\textbf{identity}(Y) = \frac{|Y|}{|E|}, Y \in \mathcal{P}(E)$$

For the **identity** semi-fuzzy quantifier, adding one element increments the result in $\frac{1}{m}$. Thus, the increase in the output obtained with the addition of elements to the argument set is linear, making possible to interpret **identity** (Y) as *'as many as possible'* or *'the more the better'*. In other way, the identity semi-fuzzy quantifier measures the relative weight of the input set Y with respect to the referential set E. That is, $\mathbf{identity}(Y) = |Y| / |E|$.

A plausible fuzzy counterpart of the identity quantifier should also produce a linear increase in the output for a linear increase in the input.

Definition 21. *[9, chapter 3] We will say that a QFM $\mathcal{F}$ fulfills the average property for the identity quantifier if:*

$$\mathcal{F}(\mathbf{identity})(X) = \frac{1}{m}\sum_{j=1}^{m}\mu_X\left(e_j\right)$$

As a result of the fulfillment of the average property for the identity quantifier, the improvement obtained in $\mathcal{F}(\mathbf{identity})(X)$ is linear with respect to the increases of the membership degrees of the argument fuzzy set. This property allows us to enquire if this intuition is translated to the fuzzy case, assuring that in the fuzzy case we will obtain a measure of the relative weight of $X \in \widetilde{\mathcal{P}}(E)$ with respect to E.

5.4.1. Analysis of the models

Models $\mathcal{F}_{owa}$, $\mathcal{F}^{MD}$, $\mathcal{F}^{I}$ and $\mathcal{F}^{A}$. Model $\mathcal{F}_{owa}$ [20, chapter 8], and models $\mathcal{F}^{MD}$, $\mathcal{F}^{I}$ [9] generalize the OWA approach, and then they trivially fulfill the property of averaging for the identity quantifier. Model $\mathcal{F}^{A}$ also fulfills this property [9],[12].

Models $\mathcal{M}$ and $\mathcal{M}_{CX}$. Models $\mathcal{M}$ and $\mathcal{M}_{CX}$ do not fulfill this property, as a direct consequence of fulfilling the propagation of fuzziness in the arguments. For $\mathcal{M}$ and $\mathcal{M}_{CX}$ models, if $\mathcal{M}(X) = a$ or $\mathcal{M}_{CX}(X) = a$, $a \geq 0.5$, then $\mathcal{M}(X'), \mathcal{M}_{CX}(X') \in [0.5, a]$ for $X' \preceq_c X$ (X' fuzzier than X). More clearly:

$$\begin{aligned}
&\mathcal{M}(\mathbf{identity})((\{1/e_1, 1/e_2, 0/e_3, 0/e_4\})) = \mathcal{M}_{CX}(\mathbf{identity})((\{1/e_1, 1/e_2, 0/e_3, 0/e_4\}))\\
&= \mathcal{M}_{CX}(\mathbf{identity})((\{0.5/e_1, 0.5/e_2, 0.5/e_3, 0.5/e_4\}))\\
&= \mathcal{M}_{CX}(\mathbf{identity})((\{1/e_1, 1/e_2, 0.5/e_3, 0.5/e_4\}))\\
&= \mathcal{M}_{CX}(\mathbf{identity})((\{0.5/e_1, 0.5/e_2, 0/e_3, 0/e_4\}))\\
&= 0.5
\end{aligned}$$

that is, as $\mathcal{M}(\mathbf{identity})((\{1/e_1, 1/e_2, 0/e_3, 0/e_4\})) = 0.5$ then for every possible X' such that $X' \preceq_c X$ the result will be at least as fuzzier as 0.5, but as 0.5 is the fuzzier possible output, $\mathcal{M}(\mathbf{identity})(X') = 0.5$.

In figure 1 we show a graphic representation of this behavior. Although we would expect a high degree of fulfillment for the identity quantifier in case 1 and a low degree in case 2, results of applying $\mathcal{M}$ or $\mathcal{M}_{CX}$ to the identity quantifier for both inputs is 0.5 and for every intermediate case between case 1) and case 2) is also 0.5.

5.5. Evaluating quantifiers over 'quantified partitions'

In this section we will analyze the behavior of the models when we simultaneously evaluate a set of fuzzy quantifiers associated to a 'quantified partition' of the quantification universe. Let us consider the set of quantification labels presented in figure 2.

For reasons we will see later on, we will restrict us to a set of labels such that $\mu_{Q_i}(x) + \mu_{Q_{i+1}}(x) = 1$ for some i. In any case, this is a very common way of dividing the reference

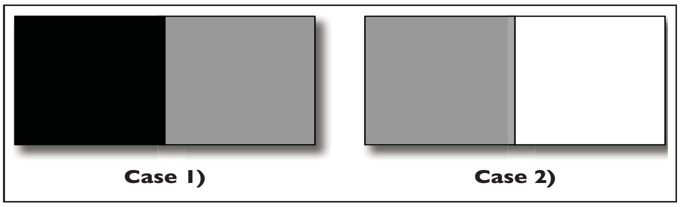

Figure 1: Indiscernible situations for the identity quantifier

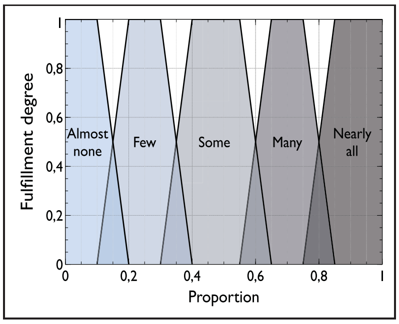

Figure 2: Partition of the quantified universe.

universe in practical applications. We will refer to quantified partitions fulfilling this property as 'Ruspini quantified partitions'.

When we consider the simultaneous evaluation of a set of quantifiers defined by means of a quantified partition, the behavior of standard DFSs ($\mathcal{M}$, $\mathcal{M}_{CX}$ $\mathcal{F}_{owa}$) and QFMs ($\mathcal{F}^A$, $\mathcal{F}^I$ and $\mathcal{F}^{MD}$) present strong differences. For some situations, models $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$ tend to produce the 0.5 output for every quantifier in the partition. Contrasting with this behavior, models $\mathcal{F}^A$, $\mathcal{F}^I$ and $\mathcal{F}^{MD}$ guarantee that the sum of the evaluation results equals 1, producing 'a distribution' of the truth between the set of quantified labels.

5.5.1. Analysis of the models

Models $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$. As a consequence of being based in trivalued cuts models $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$ present a tendency to produce 0.5 evaluation results for some situations. Let us consider a fuzzy set X such that

$$X = \{0.5/e_1, 0.5/e_2, \ldots, 0.5/e_m\}$$

then, if for a semi-fuzzy quantifier Q_i is fulfilled that there exist r, j such that $q_i(r) = 1$ and $q_i(j) = 0$ (note that every quantifier in figure 2 fulfill this property) then

$$\mathcal{M}(X) = \mathcal{M}_{CX}(X) = \mathcal{F}_{owa}(X) = 0.5$$

as it can be easily checked.

This behavior is independent of the granularity of the partition. That is, finer partitions will continue to produce a 0.5 output in each situation in which we could find some i, j such that $q_i(i) = 1$ and $q_i(j) = 0$.

Model $\mathcal{F}^A$. Before presenting the behavior of the $\mathcal{F}^A$ model, we need to introduce some definitions to precise the meaning of a 'Ruspini quantified partition'.

Definition 22. *We will say that a set of semi-fuzzy quantifiers $Q_1, \ldots, Q_r : \mathcal{P}^n(E) \to \mathbf{I}$ forms a Ruspini partition of the quantification universe if for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ it holds that*

$$Q_1(Y_1, \ldots, Y_n) + \ldots + Q_r(Y_1, \ldots, Y_n) = 1$$

Example 23. *The next set of quantifiers form a Ruspini partition of the quantification universe:*

$$Q_i(Y_1, Y_2) = \begin{cases} label_i\left(\frac{|Y_1 \cap Y_2|}{|Y_1|}\right) & Y_1 \neq \varnothing \\ \frac{1}{5} & Y_1 = \varnothing \end{cases}$$

where $label_i$ represents the 'i-th' fuzzy number in the partition. This set of fuzzy numbers forms a Ruspini partition of the quantification universe as $\sum_i Q_i(Y_1, Y_2) = 1$ for all $Y_1, Y_2 \in \mathcal{P}(E)$.

Definition 24. *[9, chapter 3]We will say that a QFM $\mathcal{F}$ fulfills the property of probabilistic interpretation of quantifiers if for all the Ruspini partitions of the quantification universe $Q_1, \ldots, Q_r : \mathcal{P}(E)^n \to \mathbf{I}$ it holds that*

$$\mathcal{F}(Q_1)(X_1, \ldots, X_n) + \ldots + \mathcal{F}(Q_r)(X_1, \ldots, X_n) = 1$$

This property is very interesting because it allows us to interpret the result of evaluating fuzzy quantified expressions as probabilities over the labels related to the quantifiers[6]. $\mathcal{F}^A, \mathcal{F}^{MD}$ and $\mathcal{F}^I$ models fulfill this property [9, chapter three], [12]. Thus, we can interpret that these models tend to distribute the truth between the set of labels of the partition, assuring that the sum of the evaluation results associated to each label adds to 1.

In addition, in [12] it has been proved that for unary quantifiers the $\mathcal{F}^A$ model tends to the Zadeh's Sigma-count model when the size of the referential E tends to infinite; that is:

$$\lim_{|E| \to \infty} \mathcal{F}^A(Q)(X) = \mu_Q\left(\frac{\sum_{e \in E} \mu_X(e)}{|E|}\right)$$

In this way, for a big $|E|$ we have $\mathcal{F}^A(Q)(X) \approx \mu_Q\left(\frac{\sum_{e \in E} \mu_X(e)}{|E|}\right)$. As $\frac{\sum_{e \in E} \mu_X(e)}{|E|}$ is a punctual value, when we apply this result to a Ruspini quantified partition like the one presented in figure 2, the weights of the evaluation of the quantified expressions tend to concentrate themselves in one quantified label q_i (being $\mathcal{F}^A(Q_i)(X) \approx 1$) or two contiguous ones q_i, q_{i+1} (being $\mathcal{F}^A(Q_i) + \mathcal{F}^A(Q_{i+1}) \approx 1$).

Models $\mathcal{F}^{MD}$ and $\mathcal{F}^I$. Models $\mathcal{F}^{MD}$ and $\mathcal{F}^I$ also fulfill the property of *probabilistic interpretation of quantifiers*. Hence, the result of evaluating a set of quantifiers $Q_1, \ldots, Q_r$ forming a Ruspini quantified partition can be interpreted as a probability defined over the quantified labels of the quantifiers. Thus, we can interpret that these models tend to distribute the weight of evaluating quantified sentences over the set of labels used to define the fuzzy quantifiers.

[6] In [22] a probabilistic interpretation of quantifiers is also used under the label semantics interpretation of fuzzy sets.

Moreover, the following result proves that in the unary case, for a 'perfectly' distributed fuzzy set, models $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ tend to assign to each quantifier a probability weight proportional to its area. Let us define an equispaced fuzzy set over [0, 1] as:

$$\mu_X(a_1) = 0 < \mu_X(a_2) = h < \mu_X(a_2) = 2h, \ldots, \mu_X(a_m) = 1$$

Then, if we restrict ourselves to piecewise continuous functions, it is fulfilled that:

$$\lim_{m\to\infty} \int_0^1 Q(X_{\geq\alpha})\,d\alpha = \lim_{m\to\infty} \mu_Q\left(\frac{1}{m}\right)h + \mu_Q\left(\frac{2}{m}\right)h + \ldots + \mu_Q\left(\frac{m-1}{m}\right)h = \int_0^1 \mu_Q(x)\,dx$$

as we are simply computing the area of the quantifier.

As a consequence, when we evaluate a set of unary quantifiers $Q_1, \ldots, Q_r : \mathcal{P}(E) \to \mathbf{I}$ over a fuzzy set following an identity function ($\mu_X(e_i) = \frac{i}{m}$) we obtain:

$$\mathcal{F}^{MD}(Q_i)(X) = \mathcal{F}^{I}(Q_i)(X) \approx area\left(\mu_{Q_i}\right)$$

This property is related to the probabilistic alpha-cut interpretation of models $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$. In this interpretation, if membership values of X are perfectly distributed, then 'weights' of the alpha cuts are perfectly distributed over the quantification universe. In this way, quantifiers of greater areas tend to 'collect' more weight than quantifiers with smaller areas. This also means that, for 'finer' quantifier partitions, weights tend to be more distributed between the quantifiers of the partition.

5.6. Fine distinction between objects

In applications of fuzzy quantifiers for ranking generation, we generally have a set of objects $o_1, \ldots, o_N$ for which the fuzzy fulfillment of a set of criteria $p_1, \ldots, p_m$ is known $X^{o_i} = \{\mu_{X^i}(p_1)/p_1, \ldots, \mu_{X^i}(p_m)/p_m\}$, where $\mu_{X^i}\left(p_j\right)/p_j$ represents the fulfillment of the criteria p_j by the object o_i. Additionally, we generally have a set of weights $W = \{\mu_W(p_1)/p_1, \ldots, \mu_W(p_m)/p_m\}$ indicating the relative importance of the criteria $p_1, \ldots, p_m$.

Fuzzy quantification can be used to generate a ranking by means of the assignment of a weight to each object, computed using an unary proportional quantified expression, $r^{o_i} = \widetilde{Q}(X^{o_i})$ when a vector of weights is not involved, or computed using a binary proportional quantified expression $r^{o_i} = \widetilde{Q}(W, X^{o_i})$ when there exists a vector of weights W to indicate the relative importance of each criteria. Hence, when we compute r^{o_i} for each $i = 1, \ldots, N$, we can rank each object with respect to *'how $\widetilde{Q}$'* criteria it fulfills (e.g., for $\widetilde{Q} =$ **many**, *'how many'*).

Fuzzy quantifiers seem specially convenient for ranking applications. As r^{o_i} indicates 'how good' is the object i in fulfilling '$\widetilde{Q}$ criteria'. We can easily adjust the quantifiers to prioritize objects fulfilling *'most of the criteria'*, *'some of them'*, *'at least 10'*, etc.

Ranking applications usually demand a great discriminative power between objects. In general, we should expect that even small variations in the inputs would produce some effect in the outputs. In order to analyze the discriminative power of QFMs, we will need some definitions:

Definition 25. *Let $h(x) : [0, 1] \to \mathbf{I}$ an strictly increasing continuous mapping; i.e., $h(x) > h(y)$ for every $x > y$. We define the unary and binary semi-fuzzy quantifiers $Q_h : \mathcal{P}(E) \to \mathbf{I}$ and*

$Q_h : \mathcal{P}(E)^2 \to \mathbf{I}$ *as*

$$Q_h(Y) = h(|Y|), Y \in \mathcal{P}(E)$$

$$Q_h(Y_1, Y_2) = \begin{cases} h\left(\frac{|Y_1 \cap Y_2|}{|Y_1|}\right) & Y_1 \neq \varnothing \\ 1 & Y_1 = \varnothing \end{cases}$$

For assuring the discriminative power of QFMs, we will require that in the case of unary quantifiers, any increase in the fulfillment of a criteria will increase $\mathcal{F}(Q_h)$. In the binary case, we will require that any increase in the fulfillment of a criteria associated with a strictly positive weight will also increase $\mathcal{F}(Q_h)$. That is, as h is strictly increasing, we expect that an increase in the values of the inputs is translated into an increase in the result of the evaluation.

Definition 26. *Let us consider $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, $i = 1, \ldots, m$, $j \in \{1, \ldots, m\}$ such that $\mu_{X_1}(e_i) = \mu_{X_2}(e_i)$, $i \neq j$, $\mu_{X_1}(e_i) < \mu_{X_2}(e_i)$, $i = j$. We say that a QFM $\mathcal{F}$ fulfills the property of discriminative ranking generation for unary quantifiers if:*

$$\mathcal{F}(Q_h)(X_2) > \mathcal{F}(Q_h)(X_1)$$

for $h(x)$ strictly increasing.

Definition 27. *Let us consider $W, X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, $i = 1, \ldots, m$, $j \in \{1, \ldots, m\}$ such that $\mu_{X_1}(e_i) = \mu_{X_2}(e_i)$, $i \neq j$, $\mu_{X_1}(e_i) < \mu_{X_2}(e_i)$, $\mu_W(i) > 0, i = j$. We say that a QFM $\mathcal{F}$ fulfills the property of discriminative ranking generation for binary quantifiers if it fulfills:*

$$\mathcal{F}(Q_h)(W, X_2) > \mathcal{F}(Q_h)(W, X_1)$$

for $h(x)$ strictly increasing.

5.6.1. Analysis of the models

Models $\mathcal{M}$ and $\mathcal{M}_{CX}$. Fulfillment of propagation of fuzziness properties makes $\mathcal{M}$ and $\mathcal{M}_{CX}$ very inconvenient for ranking applications. Examples presented in section 5.4 have shown that these models are piecewise constant, and that they are not able to differentiate between really large regions of the input space. As a consequence, these models are incapable of making fine distinction between objects.

Model $\mathcal{F}_{owa}$. Model $\mathcal{F}_{owa}$ have been presented in [20, chapter 8] as the paradigmatic example of a standard DFS non-fulfilling the properties of propagation of fuzziness. Thus, the author consider the $\mathcal{F}_{owa}$ model convenient for applications needing an 'enhanced discriminatory force'.

As $\mathcal{F}_{owa}$ model generalizes OWA, it adequately deals with the fine distinction between objects in the unary case. But in the binary case, $\mathcal{F}_{owa}$ is piecewise constant, as it proves the following example:

$$\begin{aligned} &\mathcal{F}_{owa}(Q_{id})(\{1/e_1, 1/e_2, 0.5/e_3, 0.5/e_4\}, \{1/e_1, 1/e_2, 0/e_3, 0/e_4\}) \\ &= 0.75 \\ &= \mathcal{F}_{owa}(Q_{id})(\{1/e_1, 1/e_2, 0.5/e_3, 0.5/e_4\}, \{1/e_1 1/e_2, 0.5/e_3, 0.5/e_4\}) \end{aligned}$$

In previous example, for 0.5 weights of e_3, e_4, we can modify object fulfillment in the $[0, 0.5]$ range without obtaining any difference in the output.

Model $\mathcal{F}^{MD}$. A similar problem happens with the $\mathcal{F}^{MD}$ model. As the $\mathcal{F}^{MD}$ fulfills the strong conservativity property (see [9, chapter 3]) we have

$$\mathcal{F}^{MD}(Q_{id})(W, X^{o_i}) = \mathcal{F}^{MD}(Q_{id})\left(W, W \widetilde{\cap} X^{o_i}\right)$$

and then,

$$\begin{aligned}
&\mathcal{F}^{MD}(Q_{id})(\{1/e_1, 1/e_2, 0.5/e_3, 0.5/e_4\}, \{1/e_1, 1/e_2, 1/e_3, 1/e_4\}) \\
&= \mathcal{F}^{MD}(Q_{id})(\{1/e_1, 1/e_2, 0.5/e_3, 0.5/e_4\}, \{1/e_1, 1/e_2, 0.5/e_3, 0.5/e_4\}) \\
&= 1
\end{aligned}$$

Coinciding with the $\mathcal{F}^{I}$ QFM in the unary case, $\mathcal{F}^{MD}$ model fulfills the property for unary quantifiers (see below).

Model $\mathcal{F}^{I}$. Model $\mathcal{F}^{I}$ fulfills the property of discriminative ranking generation. The proof is shown in the Apendix.

Model $\mathcal{F}^{A}$. The $\mathcal{F}^{A}$ model also fulfills the property of discriminative ranking generation. The proof is shown in the Apendix.

5.7. Some recommendations for selecting QFMs for applications

In table 2 we synthesize the behavior of the QFMs $\mathcal{F}^{MD}$, $\mathcal{F}^{I}$, $\mathcal{F}^{A}$, $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{owa}$ with respect to the set of additional criteria we have presented. We summarize some recommendations for the selection of convenient models for applications:

1. In applications that require a fine distinction between objects (e.g., ranking applications) only models $\mathcal{F}^{I}$ and $\mathcal{F}^{A}$ should be used for non unary quantifiers. In the unary case $\mathcal{F}^{MD}$ and $\mathcal{F}_{owa}$ coincide with the $\mathcal{F}^{I}$ model for increasing quantifiers, and are also acceptable.
2. In applications in which aggregative behavior is not acceptable, $\mathcal{F}^{A}$ should be avoided.
3. For maximal coherence with linguistic criteria, models $\mathcal{M}_{CX}$ and $\mathcal{F}^{A}$ are the preferred ones. Models $\mathcal{M}$ and $\mathcal{F}_{owa}$ show a good behavior as well. Model $\mathcal{F}^{I}$, being inferior to DFSs with respect to linguistic coherence, conserves linguistic transformations of the 'Aristotelian square' in the finite case.
4. If propagation of fuzziness is required, the only viable options are $\mathcal{M}$ and $\mathcal{M}_{CX}$.
5. In order to preserve the intuitions underneath the identity quantifier, guaranteeing that a linear increase in the inputs produces a linear increase in the outputs, models $\mathcal{F}^{A}$,$\mathcal{F}^{MD}$, $\mathcal{F}^{I}$ or $\mathcal{F}_{owa}$ should be selected.
6. When taken into account the behavior of QFMs over quantified partitions, if we expect more undefined results for fuzzier fuzzy sets, standard DFSs should be used. In the case of prefering that QFMs could be interpreted as probabilities over quantified labels, distributing the 'degree of fulfilment' between the different labels, the convenient models are $\mathcal{F}^{MD}$, $\mathcal{F}^{I}$ and $\mathcal{F}^{A}$.

Summing up, the model $\mathcal{F}^{A}$ is a really convenient model for all the applications in which aggregative behavior is not an impediment. $\mathcal{M}_{CX}$ is the perfect model for applications in which preservation of fuzziness properties are required, but presents the handicap that is very inadequate for ranking applications and it does not maintain the linguistic intuitions under the 'identity

	$\mathcal{F}^{MD}$	$\mathcal{F}^{I}$	$\mathcal{F}^{A}$	$\mathcal{M}$	$\mathcal{M}_{CX}$	$\mathcal{F}_{owa}$
Linguistic Compatibility	partial	partial	DFS+FAI	DFS	DFS+FAI	DFS
Aggregative behavior	No	No	Yes	No	No	No
Identity Quantifier	Yes	Yes	Yes	No	No	Yes
Propagation of Fuzziness	No	No	No	Yes	Yes	No
Quantified Partitions	Pr	Pr	Pr	Ind	Ind	Ind
Fine differentiation	No	Yes	Yes	No	No	No

Table 2: Summary of the behaviour of the QFMs. FAI: Fuzzy Argument Insertion, Pr: probability interpretation of quantified labels, Ind: tendency to 0.5 in the evaluation results over quantified labels.

quantifier'. Additionally, it has been proved that the $\mathcal{M}_{CX}$ model presents a very stable behavior [20, section 7.12], which assures a certain insensitivity against modifications in the memberships degrees.

If we need a model guaranteeing a fine distinction between objects but avoiding aggregative behavior, the best option is the $\mathcal{F}^{I}$ model. $\mathcal{F}^{I}$ also guarantees linguistic intuitions associated to the identity quantifier, allows to interpret quantified partitions as probabilities, and for fuzzy sets whose membership degrees are maximally distributed over the referential set, evaluation results provided by $\mathcal{F}^{I}$ tend to the area of the quantifier. Moreover, $\mathcal{F}^{I}$ preserves internal an external negation properties (this last property in the finite case), assuring the conservation of the linguistic relations of the 'Aristotelian square'. Although $\mathcal{F}^{I}$ is not a DFS, it is a remarkable model that presents a great equilibrium between the fulfillment of the different criteria.

$\mathcal{F}^{MD}$, sharing some of the behavior of the $\mathcal{F}^{I}$ model, is not adequate for achieving a fine differentiation of objects in the binary case. We consider linguistic behavior of $\mathcal{F}^{I}$ model superior to the linguistic behavior of $\mathcal{F}^{MD}$, as this last model does not preserve linguistic transformations of the Aristotelian square.

$\mathcal{M}$ model shares most of the behavior of the $\mathcal{M}_{CX}$ model, presenting the same problems but loosing some properties, as Fuzzy Argument Insertion.

$\mathcal{F}_{owa}$ model has been presented as the paradigmatic example of a standard DFS convenient for ranking applications, but we have seen that this model is not adequate for achieving a fine differentiation between objects with binary quantifiers. However, if we were interested in preserving the properties of standard DFSs guaranteeing some discriminative power, then the $\mathcal{F}_{owa}$ model is the convenient option.

Finally, the way in which QFMs behave over quantified partitions can guide us in our decision between standard DFSs and the remaining models. Standard DFS will tend to produce more undefined results (in the sense of closeness to $\frac{1}{2}$) for fuzzier fuzzy sets (in the sense of closeness to $\frac{1}{2}$ of their membership degrees). $\mathcal{F}^{A}$, $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ generate results that can be interpreted as probabilities, dividing the 'evaluation weight' between the different quantifiers in the partition. $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ also preserve the intuition of 'weight of the quantifier' (in the sense of the coverage of the quantification universe by the labels) for a perfect distribution of membership degrees. That is, $\mathcal{F}^{MD}$ and $\mathcal{F}^{I}$ tend to produce a result proportional to the area of the quantifier for fuzzy sets whose membership degrees tend to be equally distributed over $[0, 1]$.

6. Conclusions

In this work we have advanced in the definition of some criteria to provide a better understanding of the behavior of the most significant QFMs. First, we have compared the selected QFMs against the main set of properties presented in [20], with the objective of clarifying the differences that these models present with respect to the properties proposed in the QFM framework.

After that, we argued that previous considered properties, while being really convenient to separate 'good quantification models' from 'bad ones', are not sufficient to clearly distinguish between the set of analyzed QFMs, and specially, to help potential users in the process of selecting the most convenient model for a specific application.

In order to advance in this problem, we have introduced a new set of criteria, specially designed to differentiate the behavior of the analyzed models. An in-depth comparative analysis of the main models has been performed with respect to this new set of criteria. Based on this analysis we have established some recommendations to guide in the selection of the more adequate model.

As future work, we consider relevant the possibility of defining new oriented criteria, focused on specific types of applications.

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Apendix

Discriminative ranking generation, model $\mathcal{F}^I$

Proof. Intuitively, when we increase the fulfillment of a property $\mu_{X^{o_i}}\left(p_j\right)$ associated to a weight greater than 0 from a to b, we are adding an element to the alpha-cuts in the range $(a, b]$. As the weight of p_j is greater than 0, the relative cardinality with respect to the alpha-cuts of W containing p_j will increase.
In detail, let be $\mu_W\left(p_j\right) = c > 0$ and $\mu_{X^{o_i}}\left(p_j\right) = a$ the fulfillment of the criteria p_j for the object i. Let us consider a second fuzzy set $X^{o_i'}$ such that $\mu_{X^{o_i}}(p_z) = \mu_{X^{o_i'}}(p_z)$ for every $z \neq j$, and $\mu_{X^{o_i'}}\left(p_j\right) = b > a$.
Then,

$$\begin{aligned}
\mathcal{F}^I(Q_h)(W, X^{o_i}) &= \int_0^1 \int_0^1 Q_h\left(W_{\geq\alpha_1}, X^{o_i}_{\geq\alpha_2}\right) d\alpha_1 d\alpha_2 \\
&= \int_0^1 \int_0^a Q_h\left(W_{\geq\alpha_1}, X^{o_i}_{\geq\alpha_2}\right) d\alpha_1 d\alpha_2 + \int_0^1 \int_a^b Q_h\left(W_{\geq\alpha_1}, X^{o_i}_{\geq\alpha_2}\right) d\alpha_1 d\alpha_2 \\
&+ \int_0^1 \int_b^1 Q_h\left(W_{\geq\alpha_1}, X^{o_i}_{\geq\alpha_2}\right) d\alpha_1 d\alpha_2
\end{aligned}$$

Expressions $\int_0^1 \int_0^a Q_h\left(W_{\geq\alpha_1}, X^{o_i}_{\geq\alpha_2}\right) d\alpha_1 d\alpha_2$ and $\int_0^1 \int_b^1 Q_h\left(W_{\geq\alpha_1}, X^{o_i}_{\geq\alpha_2}\right) d\alpha_1 d\alpha_2$ are equal for o_i and $o_i^{'}$. With respect to $\int_0^1 \int_a^b Q_h\left(W_{\geq\alpha_1}, X^{o_i}_{\geq\alpha_2}\right) d\alpha_1 d\alpha_2$, for *alpha-cuts* in $(0, c] \times (a, b]$:

$$Q_h\left(W_{\geq\alpha_1}, X^{o_i}_{\geq\alpha_2}\right) < Q_h\left(W_{\geq\alpha_1}, X^{o_i'}_{\geq\alpha_2}\right)$$

as $p_j \in W_{\geq\alpha_1}$, and $p_j \in X^{o_i'}_{\geq\alpha_2}$ but $p_j \notin X^{o_i}_{\geq\alpha_2}$. And then $\mathcal{F}^I(Q_h)(W, X^{o_i}) < \mathcal{F}^I(Q_h)\left(W, X^{o_i'}\right)$. □

Discriminative ranking generation, model $\mathcal{F}^A$

Proof. □

Again, let be $\mu_W\left(p_j\right) = c > 0$ and $\mu_{X^{o_i}}\left(p_j\right) = a$ the fulfillment of the criteria p_j by the object i. Let us consider a second fuzzy set $X^{o_i'}$ such that $\mu_{X^{o_i}}(p_z) = \mu_{X^{o_i'}}(p_z)$ for every $z \neq j$, and $\mu_{X^{o_i'}}\left(p_j\right) = b > a$. We are trying to prove that:

$$\begin{aligned}
\mathcal{F}^A(Q_h)(W, X^{o_i}) &= \sum_{Y_1 \in \mathcal{P}(E)} \sum_{Y_2 \in \mathcal{P}(E)} m_W(Y_1)\, m_{X^{o_i}}(Y_1)\, Q_h(Y_1, Y_2) \\
&< \sum_{Y_1 \in \mathcal{P}(E)} \sum_{Y_2 \in \mathcal{P}(E)} m_W(Y_1)\, m_{X^{o_i'}}(Y_1)\, Q_h(Y_1, Y_2) = \mathcal{F}^A(Q_h)\left(W, X^{o_i'}\right)
\end{aligned}$$

Making some computations with $\mathcal{F}^A(Q_h)(W, X^{o_i})$ we obtain[7]:

$$\begin{aligned}
&\mathcal{F}^A(Q_h)(W, X^{o_i}) \\
&= \sum_{Y_1 \in \mathcal{P}(E)} \sum_{Y_2 \in \mathcal{P}(E)} m_W(Y_1)\, m_{X^{o_i}}(Y_2)\, Q_h(Y_1, Y_2) \\
&= \sum_{Y_1 \in \mathcal{P}(E \backslash \{p_j\})} \sum_{Y_2 \in \mathcal{P}(E \backslash \{p_j\})} (1-c)\, m_{W \backslash \{p_j\}}(Y_1)\, (1-a)\, m_{X^{o_i} \backslash \{p_j\}}(Y_2)\, Q_h(Y_1, Y_2) && (3) \\
&+ \sum_{Y_1 \in \mathcal{P}(E \backslash \{p_j\})} \sum_{Y_2 \in \mathcal{P}(E) | p_j \in Y_2} a\,(1-c)\, m_{W \backslash \{p_j\}}(Y_1)\, m_{X^{o_i} \backslash \{p_j\}}(Y_2)\, Q_h(Y_1, Y_2) && (4) \\
&+ \sum_{Y_1 \in \mathcal{P}(E) | p_j \in Y_1} \sum_{Y_2 \in \mathcal{P}(E \backslash \{p_j\})} c\,(1-a)\, m_{W \backslash \{p_j\}}(Y_1)\, m_{X^{o_i} \backslash \{p_j\}}(Y_2)\, Q_h(Y_1, Y_2) && (5) \\
&+ \sum_{Y_1 \in \mathcal{P}(E) | p_j \in Y_1} \sum_{Y_2 \in \mathcal{P}(E) | p_j \in Y_2} ca \times m_{W \backslash \{p_j\}}(Y_1)\, m_{X^{o_i} \backslash \{p_j\}}(Y_2)\, Q_h(Y_1, Y_2) && (6)
\end{aligned}$$

but if $p_j \notin Y_1$, then the relative cardinality $\frac{|Y_1 \cap C|}{|Y_1|} = \frac{|Y_1 \cap (C \cup \{p_j\})|}{|Y_1|}$ for $C \in \mathcal{P}\left(E \backslash \left\{p_j\right\}\right)$. Then, the sum of expressions 3 and 4:

$$\begin{aligned}
&\sum_{Y_1 \in \mathcal{P}(E \backslash \{p_j\})} \sum_{Y_2 \in \mathcal{P}(E) \backslash \{p_j\}} (1-c)\, m_{W \backslash \{p_j\}}(Y_1)\, (1-a)\, m_{X^{o_i} \backslash \{p_j\}}(Y_2)\, Q_h(Y_1, Y_2) \\
&+ \sum_{Y_1 \in \mathcal{P}(E \backslash \{p_j\})} \sum_{Y_2 \in \mathcal{P}(E) | p_j \in Y_2} a\,(1-c)\, m_{W \backslash \{p_j\}}(Y_1)\, m_{X^{o_i} \backslash \{p_j\}}(Y_2)\, Q_h(Y_1, Y_2) \\
&= \sum_{Y_1 \in \mathcal{P}(E \backslash \{p_j\})} \sum_{C \in \mathcal{P}(E \backslash \{p_j\})} (1-c)\, m_{W \backslash \{p_j\}}(Y_1)\, m_{X^{o_i} \backslash \{p_j\}}(C)\, Q_h(Y_1, C)
\end{aligned}$$

is not affected by the modification of $\mu_{X^{o_i}}\left(p_j\right)$; that is, it will coincide with the equivalent expression for $\mathcal{F}^A(Q_h)(W, X^{o_i}{}')$.

We will focus now in 5 and 6. For $\mathcal{F}^A(Q_h)(W, X^{o_i}{}')$, equivalent expression of 5 and 6 are obtained by substituting $(1-a)$ and a by $(1-b)$ and b, respectively; that is, we reduced $(1-a)$ by an $(b-a)$ factor and we increase a by an $(b-a)$ factor. As $h(x)$ is increasing, $(1-a)\,h(x) + ah(y) < (1-b)\,h(x) + bh(y)$ for $b > a$, $y > x$. Thus, it is trivial to see that 5 and 6 are lesser than the equivalent expressions for $\mathcal{F}^A(Q_h)(W, X^{o_i}{}')$.

[7] By $E \backslash \left\{e_j\right\}$ we denote $E \cap \overline{\left\{e_j\right\}}$; that is, the set E without the element e_j. For fuzzy sets, $X \backslash \left\{p_j\right\}$ is the projection of X eliminating the p_j element. Then, in $m_{X \backslash \{p_j\}}(Y)$ the element p_j is not taken into account in the computation of the probability mass of Y.