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Controllability of nonlinear fractional delay dynamical systems with prescribed controls

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Abstract. In this paper, we consider controllability of nonlinear fractional delay dynamical systems with prescribed controls. We firstly give the solution representation of the fractional delay dynamical systems using Laplace transform and Mittag–Leffler functions. Then we give necessary and sufficient conditions for the controllability criteria of linear fractional delay dynamical systems with prescribed controls. Further, we use a fixed point theorem to establish the sufficient condition for the controllability of nonlinear fractional delay dynamical systems with prescribed controls. In particular, we determine several sufficient conditions on the nonlinear function term so that if the linear system is controllable, then the nonlinear system is controllable. Finally, we give two examples to demonstrate the applicability of our obtained results.

Keywords: fractional delay dynamical systems, prescribed controls, controllability, Mittag–Leffler function, fixed point theorem.

1 Introduction

Fractional calculus is a generalization of integer order calculus. Unlike the integer order calculus, the fractional calculus is defined by nonlocal operators. Due to this fact, the fractional calculus has proved to be useful tools in the investigation of many phenomena with memory in engineering [32], physics [28], medicine [3], electrical circuits [1], electrodynamics of complex medium, and other fields; see, for instance, [12, 21]. In recent

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years, there has been a growing interest in investigating fractional mathematical models to improve the quality of modeling towards real world applications.

On the other hand, some authors have generalized integer order controllers to non-integer order controllers. As earlier as 1961, Manabe [27] has been devoted to fractional order systems in the area of automatic control. In 2009, Chen et al. [11] gave a clear discussion on fractional calculus as well as several known fractional order controllers and the discretisation techniques. After that, some authors began to discuss the applications of fractional calculus in control. In 2016, Ammar Soukkou et al. wrote a paper about review, design, optimization, and stability analysis of fractional-order PID controller [35]. Some pioneering works on fractional calculus in dynamic systems and controls are found in the literature (for example, see [7, 16, 18, 22, 37–39, 42]).

As one of the important topics in mathematical control theory, controllability plays an important role in the analysis and design of control systems. Controllability for various kinds of fractional differential equations has been extensively studied by many researchers [5, 6, 10, 14, 15, 17, 20, 30, 40, 41]. The above most works resort in using tools for unconstrained dynamical systems, patched with a collection of heuristic rules. However, in real world problems, many control systems have an associated set of constraints. So handling constraints in control system design is an important issue. One possible strategy for dealing with constraints is to modify the design so that limits are never violated. In the literature, the investigations of constrained controllability for both linear and nonlinear systems are numerous [2,24,25]. It is worth mentioning that Krishnan and Jayskumar [25] discussed the controllability of fractional dynamical systems with prescribed controls using Schauder's fixed point theorem.

Fractional delay dynamical systems are an important kind of fractional order systems in real life. In these years, some authors pay attention to the study about the fractional delay dynamical systems (for example, see [4, 9, 26, 34, 36]). Motivated by this literature, we propose to study the controllability for the linear fractional delay dynamical systems of the form

$$\begin{aligned} &D^{\alpha}_{0+}x(t) = Ax(t) + Bx(t-h) + Cu(t), \quad 0 < \alpha \leqslant 1,\ t \in [0,T], \\ &x(t) = \phi(t), \quad t \in [-h,0], \\ &x(T) = x_T, \\ &u(0) = u_0, \quad u(T) = u_T \end{aligned} \tag{1}$$

and the nonlinear fractional delay dynamical system

$$\begin{aligned} &D^{\alpha}_{0+}x(t) = Ax(t) + Bx(t-h) + Cu(t) + f\bigl(t, x(t), x(t-h), u(t)\bigr), \\ &\quad 0 < \alpha \leqslant 1,\ t \in [0,T], \\ &x(t) = \phi(t), \quad t \in [-h,0], \\ &x(T) = x_T, \\ &u(0) = u_0, \quad u(T) = u_T, \end{aligned} \tag{2}$$

where D^{α}_{0+} is the Caputo fractional derivative (the definition of the Caputo fractional derivative will be given in Section 2), $A, B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times m}$, the nonlinear function f

is continuous on $\mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}^m$ are state vector and control input of the systems. The above systems have initial and final conditions such that $x(0) = x_0$, $x(T) = x_T$ and $u(0) = u_0$, $u(T) = u_T$. We need to find some conditions on A, B, C, and f, which ensure that, for any given $x_0, x_T \in \mathbb{R}^n$, there exists a control $u \in \mathbb{R}^m$ with $u(0) = u_0$, $u(T) = u_T$, which produces a response $x(t; u)$ satisfying the boundary conditions $x(0; u) = x_0$ and $x(T; u) = x_T$. Here we will restrict ourselves to considering the case of controllability using continuous control functions. Hence, we will assume that the above two systems are continuous. This assumption simplifies our arguments somewhat.

This article is organized as follows. In Section 2, we briefly review some basic definitions and properties, which will be used in this paper. In Section 3, we give the solution representation of fractional delay dynamical systems using Laplace transforms and Mittag–Leffler functions. In Section 4, we give necessary and sufficient conditions for the controllability criteria of linear fractional delay dynamical systems with prescribed controls. In Section 5, we use the fixed point theorem to establish the sufficient condition for the controllability of nonlinear fractional delay dynamical systems with prescribed controls. In Section 6, the applications of the presented theory are demonstrated with two examples.

2 Preliminaries

In this section, we give some basic definitions and results that are used throughout this paper. For more details, please see [23, 31].

Definition 1. Let $[a, b]$ be a finite interval on the real axis $\mathbb{R}$. The fractional integral of order $\alpha > 0$ with the lower limit a for the function x is defined as

$$\left(I_{a^+}^{\alpha} x\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\,\mathrm{d}\tau, \quad a < t \leqslant b,$$

provided the right-hand side is pointwise defined on $[a, b]$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2. Let $[a, b]$ be a finite interval on the real axis $\mathbb{R}$, $n - 1 \leqslant \alpha < n$, $n \in N^+$, and let the function $x(t)$ have continuous derivatives up to order n such that $x^{(n)}(t)$ is absolutely continuous on $[a, b]$. The Caputo fractional derivative $(D_{a^+}^{\alpha} x)(t)$ of order α is defined as

$$\left(D_{a^+}^{\alpha} x\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau)\,\mathrm{d}\tau, \quad a < t \leqslant b.$$

The Laplace transform of the Caputo's fractional derivative $(D_{0^+}^{\alpha} x)(t)$ is

$$\mathcal{L}\left\{\left(D_{0^+}^{\alpha} x\right)(t); s\right\} = s^{\alpha} \mathcal{L}\left\{x(t); s\right\} - \sum_{i=0}^{n-1} s^{\alpha-i-1} x^{(i)}\left(0^+\right), \quad t > 0.$$

Definition 3. Three-parameter Mittag–Leffler function is defined as

$$E_{\alpha,\beta}^{\rho}(z) = \sum_{i=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta, \rho > 0,\ z \in \mathbb{R}, \tag{3}$$

where $(\rho)_k$ is the Pochhammer symbol, which is defined as $(\rho)_k = \rho(\rho+1)\cdots(\rho+k-1)$.

The Laplace transform of the three-parameter Mittag–Leffler function is

$$\mathcal{L}\bigl\{z^{\beta-1} E_{\alpha,\beta}^{\rho}\bigl(\pm a z^{\alpha}\bigr); s\bigr\} = \frac{s^{\alpha\rho-\beta}}{(s^{\alpha} \mp a)^{\rho}},$$

provided that $|as^{-\alpha}| < 1$.

An important function occurring in electrical systems is the delayed unit step function

$$u_a(t) = \begin{cases} 1, & t \geqslant a, \\ 0, & t < a, \end{cases}$$

and its Laplace transformation is given by

$$\mathcal{L}\bigl\{u_a(t); s\bigr\} = \frac{\mathrm{e}^{-as}}{s}, \quad \mathrm{Re}(s) > 0.$$

If $F(s)$ is the Laplace transformation of the function $f(t)$, i.e., $F(s) = \mathcal{L}\{f(t); s\}$, then

$$\mathcal{L}\bigl\{\mathrm{e}^{at} f(t); s\bigr\} = F(s-a)$$

and

$$\mathcal{L}\bigl\{u_a(t) f(t-a); s\bigr\} = \mathrm{e}^{-as} F(s), \quad a \geqslant 0,$$

and also we have

$$\mathcal{L}^{-1}\bigl\{\mathrm{e}^{-as} F(s); t\bigr\} = u_a(t) f(t-a), \quad a \geqslant 0. \tag{4}$$

3 Solution representation

In this section, we give the solution representation of fractional delay dynamical systems. Consider a fractional delay differential equation of the following form:

$$\begin{aligned} &D_{0+}^{\alpha} x(t) = Ax(t) + Bx(t-h) + f(t), \quad 0 < \alpha \leqslant 1,\ t \in [0, T], \\ &x(t) = \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{5}$$

where $A, B \in \mathbb{R}^{n\times n}$, $\phi(t) : [-h, 0] \to \mathbb{R}^n$ and $f : [0, T] \to \mathbb{R}^n$ are two real-value continuous functions, and $x \in \mathbb{R}^n$ is to be solved.

Following the idea in [29], we take the Laplace transform on both sides of (5) to get

$$s^{\alpha}X(s) - s^{\alpha-1}\phi(0) = AX(s) + B\int_0^{\infty} \mathrm{e}^{-st}x(t-h)\,\mathrm{d}t + F(s),$$

and by simple calculations we have

$$s^{\alpha}X(s) - s^{\alpha-1}\phi(0) = AX(s) + B\mathrm{e}^{-hs}\int_{-h}^{0} \mathrm{e}^{-s\tau}x(\tau)\,\mathrm{d}\tau + B\mathrm{e}^{-hs}X(s) + F(s),$$

where

$$X(s) = \int_0^{\infty} \mathrm{e}^{-st}x(t)\,\mathrm{d}t, \qquad F(s) = \int_0^{\infty} \mathrm{e}^{-st}f(t)\,\mathrm{d}t.$$

It follows that

$$X(s) = \frac{s^{\alpha-1}}{s^{\alpha}I - A - B\mathrm{e}^{-hs}}\phi(0) + \frac{B\mathrm{e}^{-hs}}{s^{\alpha}I - A - B\mathrm{e}^{-hs}}\int_{-h}^{0} \mathrm{e}^{-s\tau}\phi(\tau)\,\mathrm{d}\tau + \frac{F(s)}{s^{\alpha}I - A - B\mathrm{e}^{-hs}}.$$

Using Laplace inverse transform and property of the convolution, we get

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}I - A - B\mathrm{e}^{-hs}};\, t\right\}\phi(0) \\ &\quad + \mathcal{L}^{-1}\left\{\frac{B}{s^{\alpha}I - A - B\mathrm{e}^{-hs}};\, t\right\} * \mathcal{L}^{-1}\left\{\mathrm{e}^{-hs}\int_{-h}^{0} \mathrm{e}^{-s\tau}\phi(\tau)\,\mathrm{d}\tau;\, t\right\} \\ &\quad + \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}I - A - B\mathrm{e}^{-hs}};\, t\right\} * f(t). \end{aligned}$$

For brevity, we denote

$$\begin{aligned} Q_{\alpha}(t) &:= \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}I - A - B\mathrm{e}^{-hs}};\, t\right\}, \\ Q_{\alpha,\alpha}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}I - A - B\mathrm{e}^{-hs}};\, t\right\}. \end{aligned} \tag{6}$$

Define a new staircase function $p(t)$ on $[-h, \infty)$ such that

$$p(t) = \begin{cases} 0, & t \geqslant 0, \\ 1, & -h \leqslant t < 0, \end{cases} \tag{7}$$

and extend the function $\phi(t)$ to $[-h,\infty)$ such that $\phi(t)=\phi(0)$ for $t\geqslant 0$. Based on the extension, it has

$$\begin{aligned}\mathrm{e}^{-hs}\int_{-h}^{0}\mathrm{e}^{-s\tau}\phi(\tau)\,\mathrm{d}\tau &= \int_{0}^{\infty}\mathrm{e}^{-st}\phi(-h+t)p(-h+t)\,\mathrm{d}t\\ &= \mathcal{L}\bigl\{\phi(-h+t)p(-h+t);s\bigr\}.\end{aligned}$$

Hence, the solution of system (5) is given as

$$\begin{aligned}x(t)=Q_\alpha(t)\phi(0)&+\int_0^t Q_{\alpha,\alpha}(t-\tau)B\phi(\tau-h)p(\tau-h)\,\mathrm{d}\tau\\ &+\int_0^t Q_{\alpha,\alpha}(t-\tau)f(\tau)\,\mathrm{d}\tau,\end{aligned}$$

and so

$$\begin{aligned}x(t)=Q_\alpha(t)\phi(0)&+\int_{-h}^{t-h} Q_{\alpha,\alpha}(t-\tau-h)B\phi(\tau)p(\tau)\,\mathrm{d}\tau\\ &+\int_0^t Q_{\alpha,\alpha}(t-\tau)f(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Furthermore, according the definition of $p(t)$ in (7), the solution x of (5) can be written compactly as

$$x(t)=x(t;\phi)+\int_0^t Q_{\alpha,\alpha}(t-\tau)f(\tau)\,\mathrm{d}\tau, \tag{8}$$

where $x(t;\phi)$ is expressed as

$$x(t;\phi)=\begin{cases}Q_\alpha(t)\phi(0)+\int_{-h}^{t-h}Q_{\alpha,\alpha}(t-\tau-h)B\phi(\tau)\,\mathrm{d}\tau, & 0\leqslant t<h,\\ Q_\alpha(t)\phi(0)+\int_{-h}^{0}Q_{\alpha,\alpha}(t-\tau-h)B\phi(\tau)\,\mathrm{d}\tau, & h\leqslant t\leqslant T.\end{cases} \tag{9}$$

In particular, if $A=a\in\mathbb{R}$, $B=b\in\mathbb{R}$, then using (3) and (4), we can obtain

$$\begin{aligned}Q_\alpha(t)&:=\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha-a-b\mathrm{e}^{-hs}};t\right\}\\ &=\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(s^\alpha-a)(1-(s^\alpha-a)^{-1}b\mathrm{e}^{-hs})};t\right\}\\ &=\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha-a}\sum_{n=0}^{\infty}\frac{b^n\mathrm{e}^{-nhs}}{(s^\alpha-a)^n};t\right\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} b^n \mathcal{L}^{-1}\left\{ \frac{s^{\alpha-1}\mathrm{e}^{-nhs}}{(s^\alpha - a)^{n+1}}; t \right\} \\
&= \sum_{n=0}^{\infty} b^n (t-nh)^{\alpha n} E_{\alpha,\alpha n+1}^{n+1}\big(a(t-nh)^\alpha\big) u_{nh}(t) \\
&= \sum_{n=0}^{[t/h]} b^n (t-nh)^{\alpha n} E_{\alpha,\alpha n+1}^{n+1}\big(a(t-nh)^\alpha\big)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{\alpha,\alpha}(t) &:= \mathcal{L}^{-1}\left\{ \frac{1}{s^\alpha - a - b\mathrm{e}^{-hs}}; t \right\} \\
&= \mathcal{L}^{-1}\left\{ \frac{1}{(s^\alpha - a)(1-(s^\alpha-a)^{-1}b\mathrm{e}^{-hs})}; t \right\} \\
&= \mathcal{L}^{-1}\left\{ \frac{1}{s^\alpha - a} \sum_{n=0}^{\infty} \frac{b^n \mathrm{e}^{-nhs}}{(s^\alpha - a)^n}; t \right\} \\
&= \sum_{n=0}^{\infty} b^n \mathcal{L}^{-1}\left\{ \frac{\mathrm{e}^{-nhs}}{(s^\alpha - a)^{n+1}}; t \right\} \\
&= \sum_{n=0}^{\infty} b^n (t-nh)^{\alpha(n+1)-1} E_{\alpha,\alpha(n+1)}^{n+1}\big(a(t-nh)^\alpha\big) u_{nh}(t) \\
&= \sum_{n=0}^{[t/h]} b^n (t-nh)^{\alpha(n+1)-1} E_{\alpha,\alpha(n+1)}^{n+1}\big(a(t-nh)^\alpha\big).
\end{aligned}
$$

Therefore, in this case, the solution x of system (5) is given explicitly by

$$
\begin{aligned}
x(t) = x(t;\phi) + \sum_{n=0}^{[t/h]} b^n \int_0^{t-nh} &(t-\tau-nh)^{\alpha(n+1)-1} \\
&\times E_{\alpha,\alpha(n+1)}^{n+1}\big(a(t-\tau-nh)^\alpha\big) f(\tau)\,\mathrm{d}\tau,
\end{aligned}
\tag{10}
$$

where $x(t;\phi)$ is expressed as

$$
\begin{aligned}
x(t;\phi) = \sum_{n=0}^{[t/h]} b^n \Bigg(&(t-nh)^{\alpha n} E_{\alpha,\alpha n+1}^{n+1}\big(a(t-nh)^\alpha\big)\phi(0) \\
&+ b \int_{-h}^{t-(n+1)h} \big(t-\tau-(n+1)h\big)^{\alpha(n+1)-1} \\
&\qquad \times E_{\alpha,\alpha(n+1)}^{n+1}\big(a\big(t-\tau-(n+1)h\big)^\alpha\big)\phi(\tau)\,\mathrm{d}\tau \Bigg)
\end{aligned}
\tag{11a}
$$

if $0 \leqslant t < (n+1)h$,

$$x(t;\phi) = \sum_{n=0}^{[t/h]} b^n \Biggl((t-nh)^{\alpha n} E_{\alpha,\alpha n+1}^{n+1}\bigl(a(t-nh)^\alpha\bigr)\phi(0) + \sum_{n=0}^{[t/h]} b \int_{-h}^{0} \bigl(t-\tau-(n+1)h\bigr)^{\alpha(n+1)-1} \times E_{\alpha,\alpha(n+1)}^{n+1}\bigl(a\bigl(t-\tau-(n+1)h\bigr)^\alpha\bigr)\phi(\tau)\,\mathrm{d}\tau \Biggr) \tag{11b}$$

if $(n+1)h \leqslant t \leqslant T$.

With respect to representations of solutions of functional differential equations, one can refer to [8, 19, 29, 33].

4 Controllability for linear systems

Definition 4. System (1) (or (2)) is said to be controllable on $[0,T]$ if, for every given initial state ϕ and x_T, there exists a control $u \in \mathbb{R}^m$ with $u(0) = u_0, u(T) = u_T$ such that the solution of system (1) (or (2)) satisfies the boundary conditions $x(0;u) = x_0$ and $x(T;u) = x_T$.

According to (8), the solution x of system (1) can be expressed as

$$x(t) = x(t;\phi) + \int_0^t Q_{\alpha,\alpha}(t-\tau)Cu(\tau)\,\mathrm{d}\tau,$$

where $Q_\alpha(t)$, $Q_{\alpha,\alpha}(t)$, and $x(t;\phi)$ are defined as (6) and (9), respectively.

For brevity, let us denote

$$\chi(t) = \int_0^t Q_{\alpha,\alpha}(\tau)C\,\mathrm{d}\tau, \tag{12}$$

$$\Theta(t;T) = \int_{T-t}^{T} \chi^\top(\tau)\,\mathrm{d}\tau - \frac{t}{T}\int_0^T \chi^\top(\tau)\,\mathrm{d}\tau, \tag{13}$$

$$\Upsilon(t;T) = \int_0^t Q_{\alpha,\alpha}(t-\tau)C\Theta(\tau;T)\,\mathrm{d}\tau, \tag{14}$$

$$W_T = \int_0^T \chi(\tau)\chi^\top(\tau)\,\mathrm{d}\tau - \frac{1}{T}\int_0^T \chi(\tau)\,\mathrm{d}\tau \int_0^T \chi^\top(\tau)\,\mathrm{d}\tau, \tag{15}$$

where "$\top$" denotes the matrix transpose.

Define the control function of system (1) as

$$u(t) = \left(1 - \frac{t}{T}\right)u_0 + \frac{t}{T}u_T + \Theta(t;T)y(T), \tag{16}$$

where

$$y(T) = W_T^{-1}\left[x_T - x(T;\phi) - \chi(T)u_0 - \frac{1}{T}\left(\int_0^T \chi(\tau)\,\mathrm{d}\tau\right)(u_T - u_0)\right].$$

Lemma 1. *Let $u \in \mathbb{R}^m$ be defined as* (16)*. Then we have*

$$\int_0^t Q_{\alpha,\alpha}(t-\tau)Cu(\tau)\,\mathrm{d}\tau = \chi(t)u_0 + \frac{1}{T}\left(\int_0^t \chi(\tau)\,\mathrm{d}\tau\right)(u_T - u_0) + \Upsilon(t;T)y(T)$$

and $\Upsilon(T;T) = W_T$.

Proof. The idea of the proof of this lemma is exactly parallel to the proof of Lemma 4.1 in [25]. So we omit the proof. □

Theorem 1. *Assume that the matrix W_T defined in* (15) *is nonsingular. Then for an arbitrary $x_T \in \mathbb{R}^n$, the control u defined as* (16) *transfers system* (1) *from $\phi(0) \in \mathbb{R}^n$ to $x_T \in \mathbb{R}^n$ at time T with boundary conditions $u(0) = u_0$ and $u(T) = u_T$.*

Proof. Since W_T is nonsingular, the control u is well defined, and it satisfies the conditions $u(0) = u_0$ and $u(T) = u_T$. Furthermore, according to Lemma 1, we can deduce that

$$x(t) = x(t;\phi) + \chi(t)u_0 + \frac{1}{T}\left(\int_0^t \chi(\tau)\,\mathrm{d}\tau\right)(u_T - u_0) + \Upsilon(t;T)y(T).$$

It is trivial to verify that $x(0) = \phi(0)$ and $x(T) = x_T$. Thus, the control u defined as (16) transfers system (1) from $\phi(0) \in \mathbb{R}^n$ to $x_T \in \mathbb{R}^n$ at time T. That is to say, system (1) is controllable on $[0,T]$. The proof is completed. □

In fact, using the controllability of system (1), we can establish the following statement.

Theorem 2. *The system is controllable on $[0,T]$ if and only if W_T is positive definite.*

Proof. Sufficiency. Since W_T is positive definite, W_T is nonsingular. Then we can construct a control u such that it steers system (1) from the initial state $\phi(t)$ to x_T with boundary conditions $u(0) = u_0$ and $u(T) = u_T$. Thus, the system is controllable on $[0,T]$.

Necessity. Obviously, W_T is symmetric. Firstly, we prove that W_T is positive semi-definite. For any nonzero $y \in \mathbb{R}^n$, we compute the following by Cauchy–Schwartz inequality:

$$\begin{aligned} y^\top W_T y &= \int_0^T y^\top \chi(\omega)\chi^\top(\omega) y \,\mathrm{d}\omega - \frac{1}{T}\int_0^T y^\top \chi(\omega)\,\mathrm{d}\omega \int_0^T \chi^\top(\omega) y \,\mathrm{d}\omega \\ &= \int_0^T \bigl(\chi^\top(\omega)y\bigr)^\top \bigl(\chi^\top(\omega)y\bigr)\,\mathrm{d}\omega - \frac{1}{T}\Biggl(\int_0^T \chi^\top(\omega) y \,\mathrm{d}\omega\Biggr)^\top \int_0^T \chi^\top(\omega) y \,\mathrm{d}\omega \\ &= \int_0^T \bigl|\chi^\top(\omega)y\bigr|^2\,\mathrm{d}\omega - \frac{1}{T}\Biggl|\int_0^T \chi^\top(\omega) y \,\mathrm{d}\omega\Biggr|^2 \\ &\geqslant \int_0^T \bigl|\chi^\top(\omega)y\bigr|^2\,\mathrm{d}\omega - \frac{1}{T}\Biggl(\int_0^T \bigl|\chi^\top(\omega) y\bigr| \,\mathrm{d}\omega\Biggr)^2 \\ &\geqslant \int_0^T \bigl|\chi^\top(\omega)y\bigr|^2\,\mathrm{d}\omega - \int_0^T \bigl|\chi^\top(\omega)y\bigr|^2\,\mathrm{d}\omega = 0, \end{aligned}$$

which shows that the matrix W_T is positive semi-definite and the equality holds for $\chi^\top(\omega)y = \lambda \cdot 1$ for all $\omega \in [0,T]$ with the constant λ. This implies that $\chi^\top(\omega)y = 0$, $\omega \in [0,T]$. On the other hand, we consider the zero initial function $\phi = 0$ and the final input $x(T) = y$. Since the system is controllable $[0,T]$, there exists a control $u(t)$ satisfying $u(0) = u_0$ and $u(T) = u_T$ on $[0,T]$ that steers the response to $x(0) = 0$ and $x(T) = y$. It follows that

$$y = \int_0^T Q_{\alpha,\alpha}(T-\tau)Cu(\tau)\,\mathrm{d}\tau,$$

and hence,

$$y^\top y = \int_0^T y^\top Q_{\alpha,\alpha}(T-\tau)Cu(\tau)\,\mathrm{d}\tau.$$

Also, since $\chi^\top(\omega)y = 0$, $\omega \in [0,T]$, i.e.,

$$y^\top \int_0^T Q_{\alpha,\alpha}(T-\tau)C\,\mathrm{d}\tau = y^\top \chi(T) = 0,$$

therefore, $y^\top y = 0$, i.e., $y = 0$, which is a contraction for $y \neq 0$. Hence, W_T is positive definite. □

5 Controllability for nonlinear systems

In this section, we consider the controllability of system (2). According to (9), the solution x of system (2) can be expressed as

$$x(t) = x(t;\phi) + \int_0^t Q_{\alpha,\alpha}(t-\tau)Cu(\tau)\,\mathrm{d}\tau + \int_0^t Q_{\alpha,\alpha}(t-\tau)f\bigl(\tau, x(\tau), x(\tau-h), u(\tau)\bigr)\,\mathrm{d}\tau,$$

where $Q_\alpha(t)$, $Q_{\alpha,\alpha}(t)$, and $x(t;\phi)$ are defined as (6) and (9), respectively.

Using the similar arguments in Section 4, we define the control function u as

$$u(t) = \left(1 - \frac{t}{T}\right)u_0 + \frac{t}{T}u_T + \Theta(t;T)\widehat{y}(T), \tag{17}$$

where

$$\widehat{y}(T) = W_T^{-1}\Biggl[x_T - x(T;\phi) - \chi(T)u_0 - \frac{1}{T}\left(\int_0^T \chi(\tau)\,\mathrm{d}\tau\right)(u_T - u_0) - \int_0^T Q_{\alpha,\alpha}(T-\tau)f\bigl(\tau, x(\tau), x(\tau-h), u(\tau)\bigr)\,\mathrm{d}\tau\Biggr],$$

and $\chi(t)$, $\Theta(t;T)$, $\Upsilon(t;T)$, and W_T are defined in (12), (13), (14), and (15), respectively. Using the same proof of Theorem 1, we have the following lemma.

Lemma 2. *Assume that the matrix W_T is nonsingular, where W_T is defined in* (15). *Then for arbitrary $x_T \in \mathbb{R}^n$, the control function u defined in* (17) *transfers system* (2) *from x_0 to x_T at time T with $u(0) = u_0$ and $u(T) = u_T$.*

In the following, we give the sufficient condition of the controllability for the nonlinear system.

Theorem 3. *Assume that the continuous function f satisfies the following condition*:

$$\lim_{|p|\to\infty} \frac{|f(t,p)|}{|p|} = 0 \tag{18}$$

uniformly in $t \in [0,T]$, and assume that linear system (1) *is controllable on* $[0,T]$. *Then system* (2) *is controllable on* $t \in [0,T]$.

Proof. Let Q be the Banach space of all continuous functions

$$(x,u) : [-h,T] \times [0,T] \to \mathbb{R}^n \times \mathbb{R}^n$$

with the graphic norm $\|(x,u)\| = \|x\| + \|u\|$, where $\|x\| = \sup_{-h\leqslant t\leqslant T}|x(t)|$ and $\|u\| = \sup_{0\leqslant t\leqslant T}|u(t)|$. Define an operator $\mathcal{R}$ on the space Q:

$$\mathcal{R}(x,u) = (y,v),$$

where

$$y(t) = x(t;\phi) + \int_0^t Q_{\alpha,\alpha}(t-\tau)Cv(\tau)\,\mathrm{d}\tau + \int_0^t Q_{\alpha,\alpha}(t-\tau)f\bigl(\tau,x(\tau),x(\tau-h),u(\tau)\bigr)\,\mathrm{d}\tau \tag{19}$$

and

$$v(t) = \left(1-\frac{t}{T}\right)u_0 + \frac{t}{T}u_T + \Theta(t;T)\widehat{y}(T). \tag{20}$$

Thus, according to Lemma 2, one can see that the discussion of the controllability of system (2) can be reduced into the existence of the fixed point of the operator $\mathcal{R}$. We use the Schauder fixed point theorem to prove this statement.

To do this, we introduce some notations:

$$a_1 = \sup_{t\in[0,T]}\bigl|x(t;\phi)\bigr|, \qquad a_2 = \sup_{t\in[0,T]}\left|\int_0^t Q_{\alpha,\alpha}(t-\tau)C\,\mathrm{d}\tau\right|,$$

$$a_3 = \sup_{t\in[0,T]}\left|\int_0^t Q_{\alpha,\alpha}(t-\tau)\,\mathrm{d}\tau\right|, \qquad a_4 = \left|\int_0^T Q_{\alpha,\alpha}(T-\tau)\,\mathrm{d}\tau\right|,$$

$$b_1 = |u_0|, \qquad b_2 = |u_T - u_0|, \qquad b_3 = \sup_{t\in[0,T]}\bigl|\Theta(t;T)\bigr|, \qquad b_4 = \bigl|W_T^{-1}\bigr|,$$

$$b_5 = \bigl|x(T;\phi)\bigr| + \bigl|\chi(T)u_0\bigr| + \frac{1}{T}\left|\left(\int_0^T \chi(\tau)\,\mathrm{d}\tau\right)(u_T-u_0)\right|,$$

$$b = b_1 + b_2 + b_3b_4|x_T| + b_3b_4b_5, \qquad c = \max\{a_2, a_4b_3b_4, 1\},$$

$$d_1 = 6cb, \qquad d_2 = 6a_4b_3b_4c, \qquad d_3 = 6a_1, \qquad d_4 = 6a_3,$$

$$d = \max\{d_1,d_3\}, \qquad e = \max\{d_2,d_4\}.$$

Then from (19) we have

$$\begin{aligned}
\bigl|y(t)\bigr| &\leqslant a_1 + a_2\bigl|v(t)\bigr| + a_3\sup_{t\in[0,T]}\bigl|f\bigl(t,x(t),x(t-h),u(t)\bigr)\bigr| \\
&\leqslant \frac{d_3}{6} + c\bigl|v(t)\bigr| + \frac{d_4}{6}\sup_{t\in[0,T]}\bigl|f\bigl(t,x(t),x(t-h),u(t)\bigr)\bigr| \\
&\leqslant \frac{d}{6} + c\bigl|v(t)\bigr| + \frac{e}{6}\sup_{t\in[0,T]}\bigl|f\bigl(t,x(t),x(t-h),u(t)\bigr)\bigr|
\end{aligned}$$

for all $t \in [0,T]$, and from (20) it has

$$\begin{aligned}\left|v(t)\right| &\leqslant b_1 + b_2 + b_3 b_4\Big(x_T + b_5 + a_4 \sup_{t\in[0,T]}\left|f\big(t, x(t), x(t-h), u(t)\big)\right|\Big)\\ &\leqslant \frac{d_1}{6c} + \frac{d_2}{6c}\sup_{t\in[0,T]}\left|f\big(t, x(t), x(t-h), u(t)\big)\right|\\ &= \frac{1}{6c}\Big(d + e\sup_{t\in[0,T]}\left|f\big(t, x(t), x(t-h), u(t)\big)\right|\Big)\end{aligned}$$

for all $t \in [0,T]$. According to Proposition 1 in [13], for each pair of positive constants d and e, there exists a positive constant r such that if $|(x,u)| \leqslant r$, then

$$d + e\sup_{t\in[0,T]}\left|f\big(t, x(t), x(t-h), u(t)\big)\right| \leqslant r. \tag{21}$$

Based on this analysis, we take d and e as given above, and let r be chosen so that the condition in (21) is satisfied and

$$\sup_{t\in[-h,0]}\left|\phi(t)\right| \leqslant \frac{r}{3}.$$

Therefore, if

$$|x| \leqslant \frac{r}{3}, \qquad |u| \leqslant \frac{r}{3},$$

then

$$\left|(x,u)\right| = \left|x(t)\right| + \left|x(t-h)\right| + \left|u(t)\right| \leqslant r$$

for all $t \in [0,T]$. It follows that $|v(t)| \leqslant r/(6c)$. Furthermore, we have $|y(t)| \leqslant r/3$. Therefore, if let $Q(r)$ of Q as

$$Q(r) = \left\{(x,u) \in Q\colon\ \|x\| \leqslant \frac{r}{3},\ \|u\| \leqslant \frac{r}{3}\right\},$$

then $\mathcal{R}$ maps $Q(r)$ into itself.

Next, we prove the operator $\mathcal{R}$ is completely continuous. Let Q_0 be any bounded subset of $Q(r)$. Consider any sequence $\{(y_i, v_i)\}$ contained in $\mathcal{R}(Q_0)$, where we let $\mathcal{R}(x_i, u_i) = (y_i, v_i)$, $i = 1, 2, \dots$. Since f is continuous, f is uniformly bounded for all $t \in [0,T]$. It follows that $\{y_j(t)\}$ and $\{v_j(t)\}$ are uniformly bounded nd equicontinuous sequences on $[0,T]$. Then, according to Ascoli's theorem, one know that $\mathcal{R}(Q_0)$ is sequentially compact. Hence, the operator $\mathcal{R}$ is compact. Also, since $Q(r)$ is closed, bounded, and convex, the Schauder fixed point theorem implies that $\mathcal{R}$ has a fixed point $(x,u) \in Q(r)$ such that $\mathcal{R}(x,u) = (x,u)$. Therefore, the system is controllable on $[0,T]$. The proof is completed. □

6 Examples

In this section, we give two examples to demonstrate the applicability of our obtained results.

Example 1. Consider the following linear fractional delay system with delay $h = 1$:

$$\begin{aligned} &D_{0^+}^{1/2} x(t) = x(t) + x(t-1) + u(t), \quad t \in [0,T], \\ &x(t) = \phi(t), \quad t \in [-1,0], \\ &x(T) = x_T, \\ &u(0) = u_0, \quad u(T) = u_T. \end{aligned} \tag{22}$$

For this example, we consider the cases $T = 0.5$ and $T = 2$. For the case $0 \leqslant t \leqslant 0.5$, it has $[t/h] = 0$. It follows that, by (10) and (11), the solution x of (22) is given by

$$x(t) = x(t;\phi) + \int_0^t (t-\tau)^{-1/2} E_{1/2,1/2}\bigl((t-\tau)^{1/2}\bigr) u(\tau)\,\mathrm{d}\tau,$$

where

$$x(t;\phi) = E_{1/2}\bigl(t^{1/2}\bigr)\phi(0) + \int_{-h}^{t-h} (t-h-\tau)^{-1/2} E_{1/2,1/2}\bigl((t-h-\tau)^{1/2}\bigr)\phi(\tau)\,\mathrm{d}\tau.$$

In this case, $Q_{\alpha,\alpha}(t)$ is given by

$$Q_{\alpha,\alpha}(t) = t^{-1/2} E_{1/2,1/2}\bigl(t^{1/2}\bigr), \quad 0 \leqslant t \leqslant 0.5.$$

Thus, we have

$$\chi(t) = \int_0^t Q_{\alpha,\alpha}(\tau)\,\mathrm{d}\tau = t^{1/2} E_{1/2,3/2}\bigl(t^{1/2}\bigr), \quad 0 \leqslant t \leqslant 0.5,$$

$$W_T = \int_0^{1/2} \chi(\omega)\chi^\top(\omega)\,\mathrm{d}\omega - 2\int_0^{1/2} \chi(\omega)\,\mathrm{d}\omega \int_0^{1/2} \chi^\top(\omega)\,\mathrm{d}\omega = 0.1089.$$

Then, according to Theorem 2, system (22) is controllable on $[0, 0.5]$, and the control function u is defined as

$$u(t) = \left(1 - \frac{t}{T}\right) u_0 + \frac{t}{T} u_T + \Theta(t;T) y(T),$$

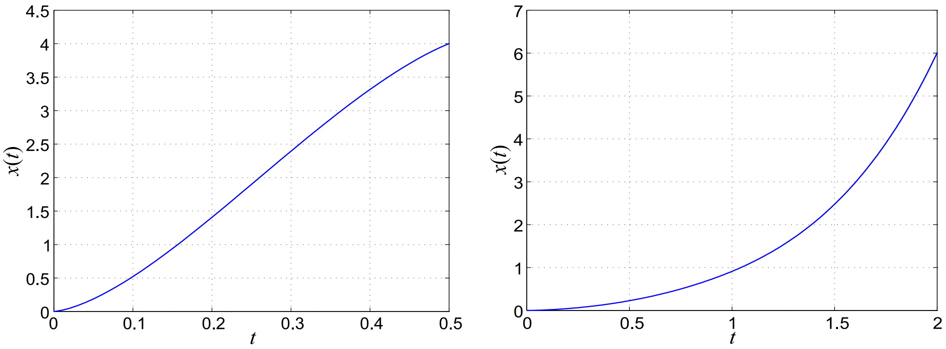
where

$$y(T) = W_T^{-1}\left[x_T - x(T;\phi) - \chi(T)u_0 - \frac{u_T - u_0}{T}\int_0^T \chi(\tau)\,\mathrm{d}\tau\right]$$

and

$$\begin{aligned} \Theta(t;T) = {} & T^{3/2} E_{1/2,5/2}\bigl(T^{1/2}\bigr) - (T-t)^{3/2} E_{1/2,5/2}\bigl((T-t)^{1/2}\bigr) \\ & - T^{1/2} t E_{1/2,5/2}\bigl(T^{1/2}\bigr) \end{aligned}$$

with $T = 0.5$.



(a) (b)

Figure 1. The curve is the state x for system (22) on the interval $[0, 0.5]$ (a); $[0, 2]$ (b).

If we choose $\phi(t) = 0$ for $t \in [-1, 0]$, $u(0) = 0$, $u(0.5) = 2$, and $x(0) = 0$, $x(0.5) = 4$, then the state x for system (22) is shown in Fig. 1(a).

For the case $T = 2$, according to (10) and (11), it has

$$Q_{\alpha,\alpha}(t) = \sum_{n=0}^{[t/h]} (t-n)^{(n+1)/2-1} E^{n+1}_{1/2,(n+1)/2}\big((t-n)^{1/2}\big), \quad t \in [0,2]$$

$$= \begin{cases} t^{-1/2} E_{1/2,1/2}(t^{1/2}), & 0 \leqslant t < 1, \\ \sum_{n=0}^{1} (t-n)^{(n+1)/2-1} E^{n+1}_{1/2,(n+1)/2}((t-n)^{1/2}), & 1 \leqslant t < 2, \\ \sum_{n=0}^{2} (t-n)^{(n+1)/2-1} E^{n+1}_{1/2,(n+1)/2}((t-n)^{1/2}), & t = 2. \end{cases}$$

Then it has

$$\chi(t) = \int_0^t Q_{\alpha,\alpha}(\tau)\,\mathrm{d}\tau, \quad t \in [0,2]$$

$$= \begin{cases} \int_0^t Q_{\alpha,\alpha}(\tau)\,\mathrm{d}\tau, & 0 \leqslant t < 1, \\ \int_0^1 Q_{\alpha,\alpha}(\tau)\,\mathrm{d}\tau + \sum_{n=0}^{1} \int_1^t (\tau-n)^{(n+1)/2-1} E^{n+1}_{1/2,(n+1)/2}((\tau-n)^{1/2})\,\mathrm{d}\tau, \\ \quad 1 \leqslant t < 2 \end{cases}$$

$$= \begin{cases} t^{1/2} E_{1/2,3/2}(t^{1/2}), & 0 \leqslant t < 1, \\ t^{1/2} E_{1/2,3/2}(t^{1/2}) + (t-1) E^{2}_{1/2,2}((t-1)^{1/2}), & 1 \leqslant t < 2, \end{cases}$$

and follows that

$$W_T = \int_0^T \chi(\omega)\chi^\top(\omega)\,\mathrm{d}\omega - \frac{1}{T}\int_0^T \chi(\omega)\,\mathrm{d}\omega \int_0^T \chi^\top(\omega)\,\mathrm{d}\omega = 48.0657 > 0.$$

Therefore, the control u is defined as

$$u(t) = \left(1 - \frac{t}{2}\right)u_0 + \frac{t}{2}u_T + \Theta(t;2)y(2), \quad t \in [0,2],$$

where

$$\Theta(t;2) = \int_{2-t}^{2} \chi^\top(\omega)\,\mathrm{d}\omega - \frac{t}{2}\int_{0}^{2} \chi^\top(\omega)\,\mathrm{d}\omega, \quad t \in [0,2],$$

$$y(2) = W_T^{-1}\left[x_T - \chi(2)u_0 - \frac{1}{2}\left(\int_{0}^{2} \chi(\tau)\,\mathrm{d}\tau\right)(u_T - u_0)\right].$$

If we choose $\phi(t) = 0$ for $t \in [-1,0]$, $u(0) = 0$, $u(2) = 1$, and $x(0) = 0$, $x(2) = 6$, then the state $x(t)$ for system (22) is shown in Fig. 1(b).

Example 2. Consider the following nonlinear fractional delay system with delay $h = 1$:

$$\begin{aligned}
&D_{0^+}^{1/2}x(t) = x(t) + x(t-1) + u(t) + t\sin x(t), \quad t \in [0,2],\\
&x(t) = 1, \quad t \in [-1,0],\\
&x(T) = x_T,\\
&u(0) = u_0, \quad u(T) = u_T.
\end{aligned}$$

According to the analysis in Example 1, one knows that the linear system

$$\begin{aligned}
&D_{0^+}^{1/2}x(t) = x(t) + x(t-1) + u(t), \quad t \in [0,2],\\
&x(t) = 1, \quad t \in [-1,0],\\
&x(T) = x_T,\\
&u(0) = u_0, \quad u(T) = u_T
\end{aligned} \tag{23}$$

is controllable on $[0,2]$. Also, the nonlinear continuous function $f = t\sin x(t)$ satisfies condition (18) in Theorem 3, hence, by Theorem 3, system (23) is controllable on $[0,2]$.

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