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GLOBAL ATTRACTION IN A SYSTEM OF DELAY DIFFERENTIAL EQUATIONS VIA COMPACT AND CONVEX SETS

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Abstract. We provide sufficient conditions for a concrete type of systems of delay differential equations (DDEs) to have a global attractor. The principal idea is based on a particular type of global attraction in difference equations in terms of nested, convex and compact sets. We prove that the solutions of the system of DDEs inherit the convergence to the equilibrium from an associated discrete dynamical system.

1. Introduction.

When modelling natural or social phenomena, stability and global attraction acquire an important role to describe their long-time behaviour. Some of these phenomena show an evolution influenced by the past states, motivating the use of delay differential equations (DDEs) [3]. If one chooses DDEs for modelling, the study of the stability properties may become a difficult task since such type of equations are related to dynamical systems with infinite-dimensional phase space. While local stability can be studied through the characteristic equation, global attraction requires different techniques. Within the study of global attraction in scalar DDEs like

$$x'(t) = -x(t) + F(x(t-\tau)), \tag{1}$$

with a unique equilibrium in the phase space, techniques coming from discrete dynamical systems can be used. The idea is to work with the related difference equation

$$x(n+1) = F(x(n)), \tag{2}$$

which "shares"[1] the equilibrium with (1). It has been proved [4, 9] that if F is continuous in an invariant interval and the equilibrium is a global attractor for (2), then it is a global attractor for (1) too. Here we use the concept of global attractor for an equilibrium in the sense that any initial state is driven to the equilibrium as time goes by. In particular, convergence to the equilibrium in (2) is related to the existence of a family of nested compact intervals containing the equilibrium in its interior and such that the image by F of any set is included in the next one. The key idea to prove that the solutions of (1) inherit the behaviour of the ones of (2) is the

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[1]The usual phase space for (1) is the set of continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}$. An equilibrium for (1) is a constant function in $[-\tau, 0]$, which can be identified with its real value.

following: the borders of such compact intervals act as "control points" the solution of (1) must consecutively pass towards the equilibrium. Therefore, the problem of finding out if a delay differential equation has a global attractor is moved to the analogous problem with difference equations, where many results are available (see [1, 5] and their references).

It is also natural to wonder if the same relation between discrete and continuous dynamics holds for the multidimensional case. In particular, consider a system of delay differential equations of the form

$$x_i'(t) = -x_i(t) + F_i(x_1(t-\tau_{i1}), \dots, x_s(t-\tau_{is})), \quad i = 1, \dots, s, \tag{3}$$

where s is a positive integer. The latter equation is a generalization to (1) while (2) is now written with $F \equiv (F_1, \dots, F_s)$ mapping a subset of $\mathbb{R}^s$ into itself. In [6, 8], the authors provide some results concerning the present problem by an extension of the ideas from the one-dimensional case to intervals in $\mathbb{R}^s$. They proved that if for every compact in a rectangular phase-space, there exists a family of nested compact intervals of $\mathbb{R}^s$ such that the image by F of each one belongs to the next interval then the equilibrium of (3) is a global attractor. In such case, the equilibrium of the difference equation is called a "strong attractor", and as its name evokes, it is a strictly stronger condition than to be a global attractor, provided $s > 1$ [6, 7]. Finally, in some cases, the existence of multidimensional strong attractors can be deduced from the study of global attraction for related scalar difference equations.

Another interesting approach within the multidimensional study is given in [2], in which a "dominance" condition in the coordinates of F is given to assure that (3) has a global attractor. This technique is employed to study well-known models as Nicholson's blowfly model with patch structure.

We would also like to mention that this kind of "discrete-continuous" relation with respect to the long-term behaviour also finds applications in the study of certain PDEs [11].

The main goal of our study is to obtain weaker conditions on the convergence to the equilibrium of the difference equation associated with (3) while keeping the inherited global attraction of the equlibrium of (3). The ideas in [6, 8] can be modified to use other "nested geometries" instead of intervals in $\mathbb{R}^s$, such as usual balls, p-norm balls or even through convex and compact sets.

In Section 2, we set some notation and definitions and include the discussion about the new concepts, reaching the statement of the main result of this work, Theorem 2.4. We also provide an example in which the results of [6] do not apply but Theorem 2.4 does. In Section 3 we write several technical lemmas concerning convex and compact sets, which become an important tool to prove the main result (Section 4).

2. **Global attraction and DDEs.**

In this section we introduce the main concept of this paper, which is related to attraction in discrete dynamical systems. Then, we compare such concept with others previously introduced, recovering several ideas of Section 2 in [6]. Finally, we state Theorem 2.4, which shows the role that the above-mentioned concept plays in the relation between attraction to an equilibrium in a DDE and in a difference equation.

Let $\mathbb{Z}^+$ and $\mathbb{R}^+$ be the set of positive integer and positive real numbers, respectively. Consider the natural numbers as $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$. In the following, we will assume that $s \in \mathbb{Z}^+$ and $D \subset \mathbb{R}^s$ is an open convex set.

We consider the system of delay differential equations

$$x_i'(t) = -x_i(t) + F_i(x_1(t-\tau_{i1}), \dots, x_s(t-\tau_{is})), \quad i = 1, \dots, s, \tag{4}$$

where $\tau_{ik} \geq 0$ and $F_i : D \to \mathbb{R}$ is locally Lipschitz-continuous, for any $i, k \in \{1, \dots, s\}$. We define $\tau = \max\{\tau_{ik} : i, k \in \{1, \dots, s\}\}$ and the function $F : D \to \mathbb{R}^s$ by

$$F(x_1, \dots, x_s) := (F_1(x_1, \dots, x_s), \dots, F_s(x_1, \dots, x_s)). \tag{5}$$

We need to work with the discrete dynamical system generated by (5). Hence, we assume that $F(D) \subset D$. We also define

$$\mathcal{C} := \mathcal{C}([-\tau, 0], \mathbb{R}^s) = \{\psi : [-\tau, 0] \to \mathbb{R}^s : \psi \text{ is continuous}\},$$

and if $B \subset \mathbb{R}^s$, we will use the following set:

$$\mathcal{C}_B := \{\psi \in \mathcal{C} : \psi([-\tau, 0]) \subset B\}.$$

For each $\phi \in \mathcal{C}$, the function

$$x(t, \phi) = (x_1(t, \phi), \dots, x_s(t, \phi))$$

will denote the unique solution of (4) with initial condition ϕ (see [3] for basic results on existence and uniqueness of solutions of DDEs). We will say that Equation (4) is the corresponding DDE to the difference equation

$$x(n+1) = F(x(n)), \quad n \in \mathbb{N}, \tag{6}$$

and viceversa. The dynamics of (4) involve the study of infinite-dimensional phase-space, whilst (6) does not. Bearing in mind this fact, the following question naturally arises: do the dynamics of the DDE (4) show any particular feature of the dynamics of its corresponding difference equation (6)? Under some circumstances, the answer is affirmative.

Since we are looking for global attraction to an equilibrium in (4), we will assume that (4) has a unique equilibrium, which will be called z_*. Thus, (6) has only one equilibrium (z_*) too.

In the scalar case ($s = 1$), if z_* is a global attractor for (6), then z_* is also a global attractor for (4) [4, 9]. Therefore, the work can be focused on techniques to assure global convergence to z_* in (6). For example, if we also assume that $F : (0, \infty) \to (0, \infty)$ is a class $\mathcal{C}^3$ function with a unique critical point (maximum), global attraction to z_* can be deduced from the local stability of z_* provided some negativity hypotheses in the Schwarzian derivative of F (see [1] and its references for results in this line).

The multidimensional case is more complicated. In reference [7], an example with $s = 2$ is provided to show that global convergence to z_* in (6) is not sufficient to guarantee the analogous property for (4). Therefore, in [6] the authors proposed a concept which recovers the principal needs of the proof in the scalar case. The key idea is that global attraction to z_* in the scalar case of (6) is equivalent to the existence of a family of compact intervals $I_n = [c_n, d_n]$, $n \in \mathbb{Z}^+$, which are nested ($F(I_n) \subset I_{n+1} \subset Int(I_n)$) and shrink around the equilibrium ($\cap_{n=1}^{\infty} I_n = z_*$). Such concept is the following one:

Definition 2.1. An equilibrium $z_* \in D$ of the system (6) is a **strong attractor** in D if for every compact set $K \subset D$, there exists a family of sets $\{I_n\}_{n \in \mathbb{Z}^+}$, where every I_n is the product of s non-empty compact real intervals, satisfying that

(C1) $K \subset Int(I_1) \subset D$,
(C2) $F(I_n) \subset I_{n+1} \subset Int(I_n)$, $\forall n \in \mathbb{Z}^+$,

(C3) $\cap_{n=1}^{\infty} I_n = z_*$.

As previously sketched, the notions of strong attractor and global attractor coincide for the scalar case (see Appendix in [6] for a proof). Finally, under this concept one can "re-link" the dynamics in the multidimensional case for both equations (4) and (6):

Theorem 2.2 (Theorem 2.5 [6])**.** *Assume that $D = (a_1, b_1) \times \cdots \times (a_s, b_s)$ and z_* is a strong attractor for (6) in D. Then, for each $\phi \in \mathcal{C}_D$, $x(t, \phi)$ is defined for any $t \geq 0$ and $\lim_{t\to\infty} x(t, \phi) = z_*$.*

Sometimes we refer to a strong attractor as an interval-strong attractor to highlight the geometry involved in the convergence to z_*.

A careful analysis of the proof of such [6, Theorem 2.5] shows that more general sets can play the role of the intervals in $\mathbb{R}^s$. We introduce the following weaker concept, which will be the key tool for the present work:

Definition 2.3. An equilibrium $z_* \in D$ of (6) is a **CC-strong attractor** in D if for every compact set $K \subset D$, there exists a family $\{K_n\}_{n\in\mathbb{Z}^+}$ of compact and convex sets with non-empty interior, satisfying that

(C1) $K \subset Int(K_1) \subset D$,
(C2) $F(K_n) \subset K_{n+1} \subset Int(K_n)$, $\forall n \in \mathbb{Z}^+$,
(C3) $\cap_{n=1}^{\infty} K_n = z_*$.

The name has been chosen to emphasize the important features of the extension of a compact interval in $\mathbb{R}$ to sets in $\mathbb{R}^s$, $s > 1$: convexity and compactness. The following example shows that the new concept includes new cases of global attractors.

Example 1. Let $D = \mathbb{R}^2$ and $H : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping aasociated to the rotation of $\frac{\pi}{4}$ radians, i.e.,

$$H(x, y) = \left(\frac{1}{\sqrt{2}}(x - y), \frac{1}{\sqrt{2}}(x + y)\right), \quad \forall (x, y) \in \mathbb{R}^2.$$

Fix $m \in (\frac{1}{\sqrt{2}}, 1)$ and let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping defined by

$$G(x, y) = (mx, my), \quad \forall (x, y) \in \mathbb{R}^2.$$

If we define $F := G \circ H$, then

$$F(x, y) = \left(\frac{m}{\sqrt{2}}(x - y), \frac{m}{\sqrt{2}}(x + y)\right), \quad \forall (x, y) \in \mathbb{R}^2.$$

First, we will prove that the origin is not a strong attractor for the system

$$(x(n + 1), y(n + 1)) = F(x(n), y(n)), \quad n \in \mathbb{N}, \tag{7}$$

in $\mathbb{R}^2$. Take any compact $K \subset \mathbb{R}^2$ and choose an arbitrary compact interval I_1 in $\mathbb{R}^2$ satisfying $I_1 \supset K$, $I_1 \ni 0$. We will write $I_1 := [c_1, c_2] \times [d_1, d_2]$ for certain $c_1, d_1 < 0$ and $c_2, d_2 > 0$. Take $q = \min\{|c_1|, |c_2|, |d_1|, |d_2|\}$. We will sketch the proof for $q = |d_1|$ (cases $q = |d_2|, |c_1|, |c_2|$ would be similarly treated). The angle between $(1, 0)$ and $(0, d_1)$ is $\frac{3\pi}{2}$. Take the point p of ∂I_1 related to an angle of $\frac{5\pi}{4} = \frac{3\pi}{2} - \frac{\pi}{4}$ with $(1, 0)$. By using that $m \in (\frac{1}{\sqrt{2}}, 1)$, it is easy to see that $p \in I_1$, but $F(p) \notin I_1$ (see Figure 1).

It is also simple to show that $(0, 0)$ is a CC-strong attractor for (7) in $\mathbb{R}^2$. For any compact $K \subset \mathbb{R}^2$, we can pick up $r_K > 0$ and a sequence of closed (2-norm) balls

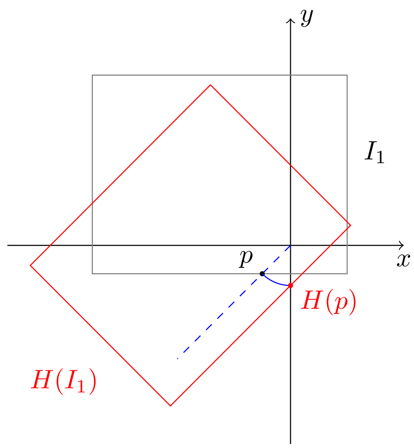

FIGURE 1. The origin is not a strong attractor.

$B_n = B[(0,0), r_K m^{n-1}]$, $n \in \mathbb{Z}^+$, such that $K \subset Int(B_1)$. The family $(B_n)_{n\in\mathbb{Z}^+}$ satisfies conditions (C1)-(C3).

Remark 1. To sum up, it is clear that every (interval-)strong attractor is a CC-strong attractor and that every CC-strong attractor is a global attractor.

These three concepts are equivalent for the scalar case. Notice that the unique convex and compact sets with nonempty interior in $\mathbb{R}$ are the non-degenerated compact intervals of $\mathbb{R}$. Hence, if $s = 1$, then

$$\text{(Interval-)strong attraction} \iff \text{CC-strong attraction} \iff \text{Global attraction.}$$

For the multidimensional case, in [7], which is an addendum of [6], the authors provide an example of a global attractor for a difference equation in $\mathbb{R}^2$ which is not a global attractor for the corresponding system of delay differential equations in $\mathbb{R}^2$. Therefore, being a strong attractor in a rectangular phase-space in $\mathbb{R}^2$ is a strictly stronger condition than being a global attractor. One can adapt the difference equation in [7] by adding independent variables driven to 0 in one iteration to see that the former condition holds for any $\mathbb{R}^s$, $s > 1$.

In addition to the former explanations, Example 1 shows that (interval-)strong attraction is a strictly stronger condition than CC-strong attraction in a phase-space contained in $\mathbb{R}^2$, even when the phase-space is rectangular. Again, if we consider independent variables driven to 0 in one iteration then the same reasoning is also valid for $\mathbb{R}^s$, $s > 1$. Thus, if $s > 1$, then

$$\text{(Interval-)strong attraction} \Longrightarrow \text{CC-strong attraction} \Longrightarrow \text{Global attraction.}$$

Now, we are ready to formulate the main result of the work.

Theorem 2.4. *Assume $F \equiv (F_1, \ldots, F_s) : D \to D$ is a continuous map and let $z_* \in D$ be a CC-strong attractor for (6) in D. Then, for each $\phi \in \mathcal{C}_D$, $x(t,\phi)$ is defined for any $t \geq 0$ and $\lim_{t\to\infty} x(t,\phi) = z_*$.*

In Section 4, we provide its proof. It requires some technical lemmas, which can be found in the following section.

Of course, finding a valid family of subsets of D to check CC-strong attraction may be quite difficult. The next corollary, based on the ideas of Example 1, provides sufficient conditions to ensure the existence of such a family of sets. Hereafter, the notation $\|\cdot\|$ will refer to a norm in $\mathbb{R}^s$.

Corollary 1. *If there exists $k \in [0,1)$ and a norm $\|\cdot\|$ in $\mathbb{R}^s$ such that*

$$\|F(x) - z_*\| \leq k\|x - z_*\|,$$

for any $x \in D$ then for each $\phi \in \mathcal{C}_D$, $x(t,\phi)$ is defined for any $t \geq 0$ and $\lim_{t\to\infty} x(t,\phi) = z_*$.

Proof. We only have to check that z_* is a CC-strong attractor for (6) in D. However, this directly comes from the following fact: the sets $A_r := \{x \in \mathbb{R}^s : \|x - z_*\| \leq r\}$, $r \geq 0$, are compact and convex. Therefore, by choosing $r_0 > 0$ such that $K \subset A_{r_0}$ the family $\{A_{r_0 k^{n-1}}\}_{n\in\mathbb{Z}^+}$ satisfies conditions (C1)-(C3). □

Based on Example 1, we provide an example of a system of delay differential equations for which we cannot use Theorem 2.5 of [6] but the general case of Theorem 2.4 applies.

Example 2. Let $\tau_{ik} \geq 0$, with $i,k \in \{1,2\}$, and consider the system

$$\begin{aligned} x'(t) &= -x(t) + \tfrac{m}{\sqrt{2}}(x(t-\tau_{11}) - y(t-\tau_{12})), \\ y'(t) &= -y(t) + \tfrac{m}{\sqrt{2}}(x(t-\tau_{21}) + y(t-\tau_{22})). \end{aligned} \tag{8}$$

Its corresponding difference equation is (7). We have checked in Example 1 that the origin is a CC-strong attractor for (6) in $\mathbb{R}^2$. Therefore, by applying Theorem 2.4, we conclude that the origin is a global attractor for the system (8).

3. Some results concerning convex sets.

In this section, we provide some lemmas that will be used in the proof of Theorem 2.4, in Section 4.

We will use the usual norm in $\mathbb{R}^s$, i.e., the norm defined by

$$\|z\|_2 = \sqrt{\sum_{j=1}^{s} {z_j}^2}.$$

for any $z \equiv (z_1, \ldots, z_s)$.

If $M \subset \mathbb{R}^s$, we will use the notation $Int(M)$ for the interior of M and ∂M for its boundary. Finally, if $v \in \mathbb{R}^s$ and $\lambda \in \mathbb{R}^+$, we will use the following notation:

$$M + v := \{x + v : x \in M\}, \quad \lambda M := \{\lambda x : x \in M\},$$
$$B_2(v,\lambda) := \{z \in \mathbb{R}^s : \|z - v\|_2 < \lambda\}.$$

Lemma 3.1. *Let $K \subset \mathbb{R}^s$ with $K \neq \mathbb{R}^s$ be a convex subset such that $Int(K)$ is non-empty. If $x \in \partial K$ and $v \in Int(K)$, then*

$$(1-\lambda)x + \lambda v \in Int(K), \quad \forall \lambda \in (0,1].$$

Moreover, there exists an $\varepsilon > 0$ such that

$$Q_{x,v,\varepsilon} := \{(1-\lambda)x + \lambda z : \ \lambda \in (0,1], \ z \in B_2(v,\varepsilon)\} \subset Int(K)$$

and $\beta_\varepsilon \in [0,1)$ such that if v_* satisfies

$$\|v_* - x\|_2 \leq \|v - x\|_2 - \varepsilon,$$
$$\cos(\angle(v_* - x, v - x)) := \frac{\langle v_* - x, v - x\rangle}{\|v_* - x\|_2 \|v - x\|_2} > \beta_\varepsilon,$$

then $v_* \in Q_{x,v,\varepsilon}$.

Proof. The first part of the lemma can be derived from Theorem 1.11 of [10].

The second part is just a direct application of the first part: if $v \in Int(K)$, then, there exists an $\varepsilon > 0$ such that $B_2(v, \varepsilon) \subset Int(K)$. Therefore, we can apply the first part of the lemma with a general $z \in B_2(v, \varepsilon)$ instead of v.

The last assertion is trivial in case $s = 1$. Otherwise, if $s \geq 2$, we pick up a plane containing x and spanned by $v - x$ and one of its orthogonal vectors. Then, a simple trigonometric reasoning (see Figure 2) will lead to see that

$$1 > \beta_\varepsilon := \frac{\sqrt{\|v-x\|_2^2 - \varepsilon^2}}{\|v-x\|_2} \geq 0.$$

□

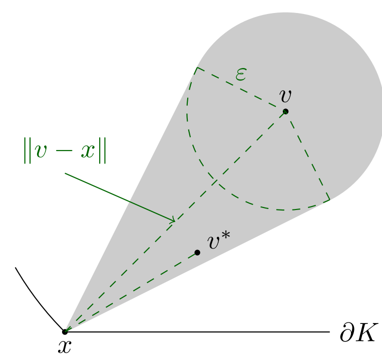

FIGURE 2. A possible set $Q_{x,v,\varepsilon}$ is represented in gray. Distances are pointed out with dashed green lines. A particular v^* satisfying the hypotheses of the last assertion of Lemma 3.1 is also depicted (color figure online).

In the following lemma, a similar assertion to the first one in Lemma 3.1 is proved: in some sense, the result is robust under small perturbations.

Lemma 3.2. *Let $K \subset \mathbb{R}^s$ with $K \neq \mathbb{R}^s$ be a convex subset such that $Int(K)$ is non-empty. If $x \in \partial K$, $v \in Int(K)$ and $g : \mathbb{R} \to \mathbb{R}^s$ is such that*

$$\lim_{\lambda \to 0} \frac{\|g(\lambda)\|}{|\lambda|} = 0,$$

then there exists $\lambda_0 > 0$ such that

$$(1-\lambda)x + \lambda v + g(\lambda) \in Int(K),$$

for any $\lambda \in (0, \lambda_0)$.

Proof. Assume that v and g satisfy the given hypotheses. We will prove this result by contradiction: we suppose that there exists a sequence of positive real numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k \to 0$ as $k \to \infty$ and

$$(1-\lambda_k)x + \lambda_k \left[v + \frac{g(\lambda_k)}{\lambda_k}\right] = (1-\lambda_k)x + \lambda_k v + g(\lambda_k) \notin Int(K),$$

for any $k \in \mathbb{N}$. On the one hand, by using the second assertion of Lemma 3.1, we conclude that for a sufficiently small $\varepsilon > 0$, we obtain

$$(1-\lambda_k)x + \lambda_k \left[v + \frac{g(\lambda_k)}{\lambda_k}\right] \notin Q_{x,v,\varepsilon}, \quad k \in \mathbb{N}.$$

Moreover, we can pick up a certain $k_0 \in \mathbb{N}$ such that

$$\|x + \lambda_k(v - x) + g(\lambda_k) - x\|_2 = \|\lambda_k(v - x) + g(\lambda_k)\|_2 < \|v - x\|_2 - \varepsilon, \quad k \geq k_0,$$

and by the last assertion of the same lemma, there exists a $\beta_\varepsilon \in [0, 1)$ such that

$$\cos(\angle(\lambda_k(v - x) + g(\lambda_k), v - x)) \leq \beta_\varepsilon < 1, \quad k \geq k_0. \tag{9}$$

On the other hand,

$$\begin{aligned}\lim_{k\to\infty} \cos(\angle(\lambda_k(v - x) + g(\lambda_k), v - x)) &= \lim_{k\to\infty} \frac{\langle \lambda_k(v - x) + g(\lambda_k), v - x\rangle}{\|\lambda_k(v - x) + g(\lambda_k)\|_2\|v - x\|_2} \\ &= \lim_{k\to\infty} \frac{\left\langle v - x + \frac{g(\lambda_k)}{\lambda_k}, v - x\right\rangle}{\|v - x + \frac{g(\lambda_k)}{\lambda_k}\|_2\|v - x\|_2} = 1. \end{aligned} \tag{10}$$

By (9) and (10), we obtain a contradiction. □

We also will need the following result, which can be easily proved.

Lemma 3.3. *If $v \in \mathbb{R}^s$, $\lambda \geq 0$ and $M, W \subset \mathbb{R}^s$ are convex, then the sets $M + v$, $M \cap W$ and λM are also convex. Moreover, if $0 \in M$ and $\lambda < \mu$ then $\lambda M \subset \mu Int(M)$.*

4. **Proof of Theorem 2.4.** Before giving the proof, we present the notation we will need. For each $t \geq 0$ in the maximal interval of the solution $x(t, \phi)$ of (4), we will define x_t^ϕ as the element of $\mathcal{C}$ such that

$$x_t^\phi(\theta) = x(t + \theta, \phi), \quad \theta \in [-\tau, 0].$$

We also define the function $f : \mathcal{C} \to \mathbb{R}^s$ by

$$f(\psi) = (F_1(\psi_1(-\tau_{11}), \dots, \psi_s(-\tau_{1s})), \dots, F_s(\psi_1(-\tau_{s1}), \dots, \psi_s(-\tau_{ss}))),$$

for any $\psi \equiv (\psi_1, \dots, \psi_s) \in \mathcal{C}$. Then, we can rewrite (4) as

$$x'(t) = -x(t) + f(x_t). \tag{11}$$

The ω-limit set of ϕ (e.g. see [3, Chapter 4]) will be denoted as $\omega(\phi)$.

The proof is based in the one for Theorem 2.5 in [6].

Proof. Fix an arbitrary function ϕ as in the statement of the theorem. Since z_* is a CC-strong attractor for (6) in D, if we choose the compact set

$$K = \{\phi(t) : t \in [-\tau, 0]\},$$

we can pick up a family $\{K_n\}_{n\in\mathbb{Z}^+}$ of compact and convex sets with non-empty interior satisfying conditions (C1)-(C3). First we prove that

$$x(t, \phi) \in K_1, \quad \forall t \geq 0. \tag{12}$$

We remark that the right-hand side of (11) is completely continuous and Theorem 3.2 of [3, Chapter 3] asserts that if $x(t, \phi)$ is not defined for all $t \geq 0$, then it must leave each closed bounded set of D, in particular, K_1. Then we will verify that $x(t, \phi)$ cannot leave K_1, so (12) would be true. First, notice that $x(t, \phi) = \phi(t) \in K \subset Int(K_1)$, for any $t \in [-\tau, 0]$. We will proceed by contradiction: we will suppose that there exists a first time $T > 0$ such that $x(T, \phi) \in \partial K_1$ and $x(T + \lambda, \phi) \notin K_1$, $\lambda \in (0, \lambda_*)$, for some $\lambda_* > 0$. The solution $x(t, \phi)$ is of class $\mathcal{C}^1$ for $t > 0$ and therefore, there exists a continuous function g such that

$$x(T + \lambda, \phi) = x(T, \phi) + \lambda x'(T, \phi) + g(\lambda) \tag{13}$$

and

$$\lim_{\lambda\to 0}\frac{\|g(\lambda)\|}{|\lambda|}=0.$$

We can use (11) in (13) and obtain

$$x(T+\lambda,\phi)=x(T,\phi)+\lambda(f(x_T^\phi)-x(T,\phi))+g(\lambda).$$

Since $x(T,\phi)\in\partial K_1$ and $f(x_T^\phi)\in Int(K_1)$, a direct application of Lemma 3.2 gives us that $x(T+\lambda)\in\ Int(K_1)$ for sufficiently small $\lambda>0$. Therefore, condition (12) is true.

Now we prove that there is a time $t_1>0$ such that

$$x(t,\phi)\in K_2,\quad t\geq t_1. \tag{14}$$

First, we prove that if $x(T^*,\phi)\in K_2$, for some $T^*\geq 0$, then $x(t,\phi)\in K_2$, for $t\geq T^*$. Suppose there exists such $T^*\geq 0$ such that $x(T^*,\phi)\in K_2$ and $T'>T^*$ satisfying

$$x(T',\phi)\notin K_2. \tag{15}$$

Then, since the solution is continuous, $x(t,\phi)$ must intersect ∂K_2 at least, in some T'' such that $T^*\leq T''<T'$. Take the maximum t when this happens and rename it as T''. Again, as an application of Lemma 3.2, we get that $x(T''+\lambda)\in\ Int(K_2)$, for a sufficiently small $\lambda>0$. Then, $x(t,\phi)$ must belong to K_2 at $t=T'$ because it cannot intersect the boundary once more between T'' and T'.

Finally, we prove that there exists $t_1\geq 0$ so that the assertion related to (14) holds. Suppose that $x(t,\phi)\notin K_2$, $\forall t\geq 0$. Define the sets (see Figure 3)

$$K_{2,\mu}:=\{z_*+\mu(x-z_*):\ x\in K_2\},\quad \mu\geq 1.$$

By consecutive applications of Lemma 3.3, it is clear that, K_2-z_*, $\mu(K_2-z_*)$ and, finally, $K_{2,\mu}$ are convex sets. Moreover, by applying the same lemma, we use that $0\in\mu(K_2-z_*)$ to write $K_{2,\mu_1}\subset Int(K_{2,\mu})$ if $\mu_1<\mu$. Then if we set $G_\mu=K_1\cap K_{2,\mu}$, which is a convex set (Lemma 3.3), we obtain the following chain of inclusions:

$$F(G_\mu)\subset F(K_1)\subset K_2\subset Int(K_1)\cap Int(K_{2,\mu})=Int(G_\mu)\subset G_\mu,$$

which holds for every $\mu>1$.

The condition $x(t,\phi)\notin K_2$, $\forall t\geq 0$ is equivalent to the assumption that $x(t,\phi)\in\partial G_\mu$, with $\mu(=\mu(t))>1$, for any $t\geq 0$. Pick up an arbitrary $t_*\geq 0$, then

$$x(t_*+\lambda,\phi)=x(t_*,\phi)+\lambda x'(t_*,\phi)+g^*(\lambda)=x(t_*,\phi)+\lambda(f(x_{t_*}^\phi)-x(t_*,\phi))+g^*(\lambda),$$

where

$$\lim_{\lambda\to 0}\frac{\|g^*(\lambda)\|}{|\lambda|}=0.$$

Reasoning in a similar way to what we have done before and using Lemma 3.2, there exists $\lambda_{t_*}>0$ such that $x(t_*+\lambda,\phi)\in Int(G_{\mu(t_*)})$, for $\lambda\in(0,\lambda_{t_*})$. This situation happens again if $x(t,\phi)$ eventually returns to $\partial G_{\mu(t_*)}$ in another $t_{**}>t_*$. Hence, $\mu(t_*)\geq\mu(t_*+\lambda)$, for every $\lambda\in\mathbb{R}^+$, and since the reasoning is valid for any $t_*\geq 0$, the function μ is nonincreasing in $[0,\infty)$. Moreover, by our assumption, μ is bounded from below by 1, which implies $\mu(t)\to r\in[1,\mu(0)]$ and $x(t,\phi)\to\partial G_r$ as $t\to\infty$.

By an application of Lemma 1.4 and Lemma 2.1 of [3, Chapter 4], $\omega(\phi)$ is nonempty and invariant. By previous reasonings, $\omega(\phi)\subset\mathcal{C}_{\partial G_r}$. Nevertheless, a nonempty subset of $\mathcal{C}_{\partial G_r}$, $r\geq 1$, cannot be invariant. In fact, if $x\equiv x(t,\phi)\in\partial G_r$,

$x(t+\lambda, \phi)$ belongs to $Int(G_r)$ for any $\lambda \in (0, \lambda_0)$, for a sufficiently small λ_0. Obviously, this is a contradiction and the assertion related to (14) is true.

We can repeat the previous procedure replacing K_1 by K_2 and K_2 by K_3, and so on. We have an inductive argument, obtaining an increasing sequence of non-negative real numbers $\{t_n\}_{n\in\mathbb{N}}$ satisfying $x(t, \phi) \in K_{n+1}$, $\forall t \geq t_n$. Since, by (C3), $\cap_{n=1}^{\infty} K_n = z_*$, it follows that $\lim_{t\to\infty} x(t, \phi) = z_*$.

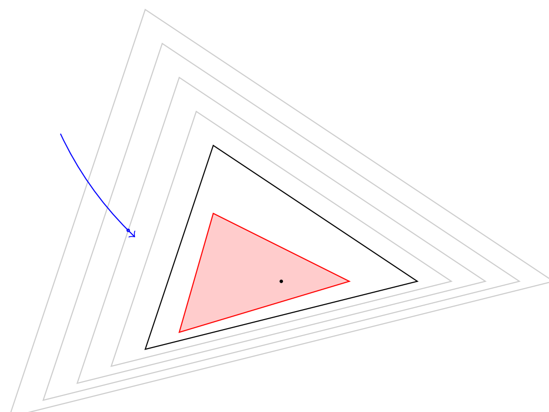

FIGURE 3. Possible behaviour of $x(t, \phi)$ (blue). The boundary of the set K_2 is represented in black. The boundaries of $K_{2,\mu}$, for some values $\mu > 1$ are represented in grey. The blue arrow represents $x'(t, \phi)$, which "points to the interior" of a $K_{2,\mu}$. $f(K_2)$ is represented in red. The equilibrium z_* is depicted as a point inside $f(K_2)$ (color figure online).

□

Remark 2. If the elements of the family $\{K_n\}_{n\in\mathbb{Z}^+}$ satisfying the conditions of CC-strong attraction are balls with respect to the p-norm, $p > 1$, in $\mathbb{R}^s$ (not necessarily with the same center, but "shrinking" around the strong attractor) then the proof of Theorem 2.4 can be obtained in an easier way. The idea is that the sets K_n have "regular boundary" and we can use the discrete Hölder's inequality to prove that the solution enters in each K_n. Thus, we avoid working with general convex sets.

Remark 3. A result of independent interest for positive invariance in Equation (4) can be derived from the arguments we used in the proof of Theorem 2.4: if $M \subset D$ is compact, convex and $F(M) \subset Int(M)$, then if $\phi \in \mathcal{C}_M$, the solution $x(t, \phi)$ of (4) will remain in M, for $t \geq 0$.

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